VIBRATIONS

OF A DEEP BEAM

WITH

CENTRALLY ATTACHED MASS

A THESIS
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NOMENCLATURE

The following nomenclature is used in this paper:

E = modulus of elasticity

G = modulus of rigidity

I = area moment of inertia of cross section

A = cross-sectional area

γ = weight per unit volume

k = numerical shape factor for cross-section

y = total deflection

ψ = bending slope

Y = normal function of y

Ψ = normal function of ψ

ξ = non dimensional length of beam (x/L)

i = $\sqrt{-1}$

p = angular frequency

L = length of beam

$b^2 = \frac{1}{EI} \cdot \frac{\gamma A}{g} \cdot L^4 \cdot p^2$

$r^2 = I/AL^2$

$s^2 = EI/kAGL^2$

g = acceleration due to gravity

M = mass of the attached mass

B = moment of inertia of mass

m = mass of the beam

$\lambda = \alpha/\beta$

$\lambda' = \alpha'/\beta$

$\zeta = \frac{\beta^2 - s^2}{\alpha^2 + s^2} = \frac{\alpha^2 + r^2}{\alpha^2 + s^2} = \frac{\beta^2 - s^2}{\beta^2 - r^2} = \frac{\alpha^2 + r^2}{\beta^2 - r^2}$

p_o = frequency from classical theory

j_o = the torsional spring constant of the end elastically restrained against rotation.

INTRODUCTION

The effect of moment of inertia of an attached mass to a beam, on the natural frequencies of the beam-mass system has received considerable attention by several investigators. Nowacki (1)* and Prescott (2) have discussed the case of the transverse vibration of a cantilever beam carrying a concentrated mass at one end. Both have ignored the moment of inertia of the attached mass. Nowacki has also considered the case of a simply supported beam carrying a mass at the center, wherein moment of inertia of mass is again neglected. It is stated wrongly that due to symmetry only symmetric modes exist. Vibrations of beams carrying a mass having moment of inertia has been discussed in recent years by J.C. Maltbaek (3), Yu Chen (4), W.E. Baker (5), and M.S. Hess (6). In all these cases, the mass is placed at the middle of the beam having symmetric boundary conditions. Vibrations of a beam with mass at any arbitrary position is studied by Y.C. Das and L.S. Srinath (7).

But in all these cases the effect of rotatory inertia and shear deformation of the beam, on the frequencies and normal modes of the composite system is neglected. All these authors studied this problem in the light of the classical one-dimensional Bernoulli-Euler theory which

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*Numbers in parantheses refer to the list of references at the end of the paper.

considers only the deflection of the beam due to flexure and the inertia forces due to transverse acceleration. But this theory is inadequate for the study of higher modes of beams, as well as for the modes of beams for which the cross-sectional dimensions are not small compared to their lengths between nodal sections.

The first correction to the classical theory of beam was made by Lord Rayleigh (8). The elements of a vibrating beam perform not only a translatory but also rotatory motion. Rayleigh recognized this additional inertia load and showed its effects on the response of a vibrating beam. This effect is known as the effect of rotatory inertia.

In 1921 Timoshenko (9, 10) showed that a more refined analysis is possible if the beam deflection due to shear in addition to the rotatory inertia be taken into account. This modified Timoshenko theory substantially agreed with experimental results. On the other hand, the exact equations, due to Pochhammer (11) and to Chree (12), have been derived from the general equations of the theory of elasticity. The resulting frequency equation for flexural vibrations is discussed by Bancroft (13) and the necessary computations are carried out by Hudson (14) and Davis (15). It is seen that the results from Timoshenko's equation are in remarkably good agreement with those obtained by Hudson from the exact elasticity equations.

Since then there has been considerable research interest in applying the Timoshenko theory to the transient responses of beams as well as the free and

forced vibrations. Anderson (16) and Dolph (17), in dealing with this problem, gave general solutions and complete analysis of uniform hinged-hinged beam. Using methods of Ritz and Galerkin, Haung (18) also presented the results for a hinged-hinged beam. Earlier Kruszewski (19) obtained frequency equations for cantilever and free-free beams by solving a complete differential equation in deflection with prescribed homogeneous boundary conditions. Haung (20) derived the frequency equations and normal modes of free vibrations of finite beams including the effect of shear and rotatory inertia for various simple end conditions.

The aim of this ~~paper~~ is to study the vibrations of a beam with central attached mass having a finite moment of inertia, and including the effects of rotatory inertia and shear deformation of the beam. To achieve this, the following novel features are used:

i) The solutions are obtained for two differential equations in total deflection and bending slope, respectively.

ii) The constants in these solutions are related by any one of the two original coupled equations from which the foregoing two complete differential equations are derived.

iii) The boundary conditions prescribed are homogeneous.

iv) Symmetry and anti-symmetry properties are considered.

Due to symmetry of the boundary conditions of the beam and the location of mass, two types of modes, viz. symmetric and asymmetric, exist. By using this property, only half of the beam is considered to obtain the frequency

equations. In case of symmetric modes there is only translatory displacement of the attached mass in transverse direction, and in case of asymmetric modes there is only rotation of the attached mass. These properties are used in setting up the proper conditions at the center of the beam.

The following types of boundary conditions, which are of general interest, are considered:

i) Supported - Supported,

ii) Clamped - Clamped,

iii) Free - Free,

iv) Elastically restrained against rotation at both ends.

Various special cases are obtained from the general frequency equation of the beam-mass system. These special cases check with the results obtained by other authors.

As the general characteristic equations of the beam-mass system are highly transcendental, they are solved with the help of IBM - 1620 computer to get the first two frequencies in each case.

This problem has lot of bearing to practical situations. A machine resting on a deep beam or the wing spar section of an aircraft may be idealised to one of the above problems.

SECTION I

ANALYSIS OF THE PROBLEM

A. DIFFERENTIAL EQUATIONS:

The coupled equations for the total deflection y and the bending slope ψ, as derived by Timoshenko (21) are,

$$EI\frac{\partial^2\psi}{\partial x^2} + k\left(\frac{\partial y}{\partial x} - \psi\right)AG - \frac{I\gamma}{g}\cdot\frac{\partial^2\psi}{\partial t^2} = 0 \qquad (1.1)$$

$$\frac{\gamma A}{g}\frac{\partial^2 y}{\partial t^2} - k\left(\frac{\partial^2 y}{\partial x^2} - \frac{\partial\psi}{\partial x}\right)AG = 0 \qquad (1.2)$$

Eliminating ψ and y from equations (1.1) and (1.2), the following two uncoupled differential equations in y and ψ are obtained:

$$EI\frac{\partial^4 y}{\partial x^4} + \frac{\gamma A}{g}\frac{\partial^2 y}{\partial t^2} - \left(\frac{\gamma I}{g} + \frac{EI}{gk}\cdot\frac{\gamma}{G}\right)\frac{\partial^4 y}{\partial x^2\partial t^2} + \frac{\gamma I}{g}\cdot\frac{\gamma}{gkG}\frac{\partial^4 y}{\partial t^4} = 0 \qquad (1.3)$$

$$EI\frac{\partial^4\psi}{\partial x^4} + \frac{\gamma A}{g}\frac{\partial^2\psi}{\partial t^2} - \left(\frac{\gamma I}{g} + \frac{EI}{gk}\cdot\frac{\gamma}{G}\right)\frac{\partial^4\psi}{\partial x^2\partial t^2} + \frac{\gamma I}{g}\cdot\frac{\gamma}{gkG}\cdot\frac{\partial^4\psi}{\partial t^4} = 0 \qquad (1.4)$$

The first two terms of equation (1.3) constitute the classical, Bernoulli - Euler, equation; the two terms containing k in the denominator arise from the inclusion of shear deflection and the remaining member of the left side is rotatory inertia term.

The shear slope, moment and shear are given by:

Shear slope; $\phi(x,t) = \frac{\partial y}{\partial x} - \psi$ (1.5)

Moment; $M(x,t) = -EI\frac{\partial\psi}{\partial x}$ (1.6)

Shear; $Q(x,t) = kAG(\frac{\partial y}{\partial x} - \psi)$ (1.7)

For the simplest end configurations, the boundary conditions are the following:

Hinged End; $y = 0$ and $\frac{\partial\psi}{\partial x} = 0$ (1.8)

Clamped End; $y = 0$ and $\psi = 0$ (1.9)

Free End; $\frac{\partial\psi}{\partial x} = 0$ and $\frac{\partial y}{\partial x} - \psi = 0$ (1.10)

Elastically restrained against rotation;

$y = 0$ and $EI\frac{\partial\psi}{\partial x} = j_o\psi$ (1.11)

B. SOLUTIONS:

Let us take the solutions of equations (1.1) to (1.4) in the form

$$y\,(x,\ t) = Y\,(\xi)\,e^{ipt} \tag{1.12}$$

$$\psi\,(x,\ t) = \Psi(\xi)\,e^{ipt} \tag{1.13}$$

Substituting these solutions into equations (1.1) to (1.4) and omitting the factor e^{ipt}, equations (1.1) to (1.4) are reduced to

$$s^2\Psi'' - (1 - b^2 r^2 s^2)\Psi + \frac{Y'}{L} = 0 \tag{1.14}$$

$$Y'' + b^2 s^2 Y - L\Psi' = 0 \tag{1.15}$$

$$Y^{iv} + b^2(r^2+s^2)Y'' - b^2(1 - b^2 r^2 s^2)\,Y = 0 \tag{1.16}$$

$$\Psi^{iv} + b^2(r^2+s^2)\Psi'' - b^2(1 - b^2 r^2 s^2)\Psi = 0 \tag{1.17}$$

The dimensionless parameter b is directly related to frequencies of vibration, p. The dimensionless

*Prime indicates the derivative with respect to ξ

parameters r, and s are measures of the effects of rotatory inertia, and shear deformation, respectively.

There are two sets of solutions of (1.16) and (1.17).

CASE I: When $\left[(r^2-s^2)^2+4/b^2\right]^{1/2} > (r^2+s^2)$,

the solutions of equations (1.16) and (1.17) can be found as

$$Y = C_1 \cosh b\alpha\xi + C_2 \sinh b\alpha\xi + C_3 \cos b\beta\xi + C_4 \sin b\beta\xi \tag{1.18}$$

$$\Psi = C_1' \sinh b\alpha\xi + C_2' \cosh b\alpha\xi + C_3' \sin b\beta\xi + C_4' \cos b\beta\xi \tag{1.19}$$

where

$$\alpha = \frac{1}{\sqrt{2}}\left[-(r^2+s^2)+\left\{(r^2-s^2)^2+\frac{4}{b^2}\right\}^{1/2}\right]^{1/2}$$

$$\beta = \frac{1}{\sqrt{2}}\left[(r^2+s^2)+\left\{(r^2-s^2)^2+\frac{4}{b^2}\right\}^{1/2}\right]^{1/2}$$

CASE II: When $\left[(r^2-s^2)^2+\frac{4}{b^2}\right]^{1/2} < (r^2+s^2)$,

the solutions of equations (1.16) and (1.17) are

$$Y = C_1 \cos b\alpha'\xi + i C_2 \sin b\alpha'\xi + C_3 \cos b\beta\xi + C_4 \sin b\beta\xi \tag{1.20}$$

$$\Psi = i C_1' \sin b\alpha'\xi + C_2' \sin b\alpha'\xi + C_3' \sin b\beta\xi + C_4' \cos b\beta\xi \tag{1.21}$$

where $\alpha' = -i\alpha$

It is but natural that the solutions of (1.18) and (1.19), or (1.20) and (1.21) are the solutions of the original coupled equations (1.14) and (1.15).

Only one half of the constants in equations (1.18) and (1.19) or (1.20) and (1.21) are independent.

They are related by the equations (1.14) and (1.15) as follows:

$$C_1' = \frac{b}{L} \cdot \frac{\alpha^2 + s^2}{\alpha} \cdot C_1 \tag{1.21}$$

$$C_2' = \frac{b}{L} \cdot \frac{\alpha^2 + s^2}{\alpha} \cdot C_2 \tag{1.22}$$

$$C_3' = -\frac{b}{L} \cdot \frac{\beta^2 - s^2}{\beta} \cdot C_3 \tag{1.23}$$

$$C_4' = \frac{b}{L} \cdot \frac{\beta^2 - s^2}{\beta} \cdot C_4 \tag{1.24}$$

C. FREQUENCY EQUATIONS:

The application of appropriate boundary conditions, the continuity equations at the center and relations of integration constants (1.21) to (1.24) to equations (1.18) and (1.19) or (1.20) and (1.21) yields for each type of beam a set of four homogeneous linear algebraic equations in four constants C_1 to C_4 with or without primes. In order to have the non-trivial solution, the determinant of the coefficients of C_s must be equal to zero. This leads to the frequency equation in each case from which the natural frequencies can be determined.

Since the boundary conditions and the location of the mass on the beam are symmetric, instead of writing two set of boundary conditions and continuity equations in the center, we can cut the beam-mass system into two similar parts. In this way, we have to write only one set of boundary conditions and the continuity equations. This simplifies to solving only one half of the beam with four conditions. In case of symmetric modes, the

slope will be zero always so that the mass can only translate, but cannot rotate. Then, the shear force will be proportional to the inertia force of the mass. In case of anti-symmetric modes, the deflection will be zero always so that the mass can only rotate, but cannot translate. Then the moment is proportional to the inertia moment (product of mass moment of inertia and angular acceleration) of the mass. Thus consideration of symmetry and anti-symmetry properties gives rise very simple continuity equations.

SECTION II

SIMPLY SUPPORTED BEAM WITH CENTRAL MASS

A. SYMMETRIC MODES:

The boundary and continuity conditions can be written for the supported beam shown schematically in fig. 2-1. These conditions are:

$$\text{at } x = 0, \quad \begin{cases} y = 0 \\ \partial\psi/\partial x = 0 \end{cases}$$
$$\text{at } x = L/2, \quad \begin{cases} \psi = 0 \\ Q = kAG\left(\frac{\partial y}{\partial x} - \psi\right) = -\frac{M}{2}\cdot\frac{\partial^2 y}{\partial t^2} \end{cases} \tag{2.1}$$

If equations (1.18) and (1.19) or equations (1.20) and (1.21) are substituted in (2.1), the frequency equations are generated from the requirement that not all of the constants C_s can be zero. Frequency equations are:

1) When $\left[(r^2-s^2)^2+\frac{4}{b^2}\right]^{1/2} > (r^2+s^2)$, frequency equation is

$$(1+\xi) = \frac{M}{m}\cdot\frac{b}{2}\left\{\beta\tan\frac{b\beta}{2} - \alpha\xi\tanh\frac{b\alpha}{2}\right\} \tag{2.2}$$

LIMITING CASES :

a) Ignoring the effect of shear deformation and rotatory inertia, we obtain

$$\alpha = \beta = 1/\sqrt{b} \qquad \text{since} \quad r = s = 0 \tag{2.3}$$

Frequency equation (2.2) takes the following form by using condition (2.3):

$$2 = \frac{M}{m}\cdot\frac{\sqrt{b}}{2}\left[\tan\frac{\sqrt{b}}{2} - \tanh\frac{\sqrt{b}}{2}\right] \tag{2.4}$$

Equation (2.4) is same as derived by Baker (5).

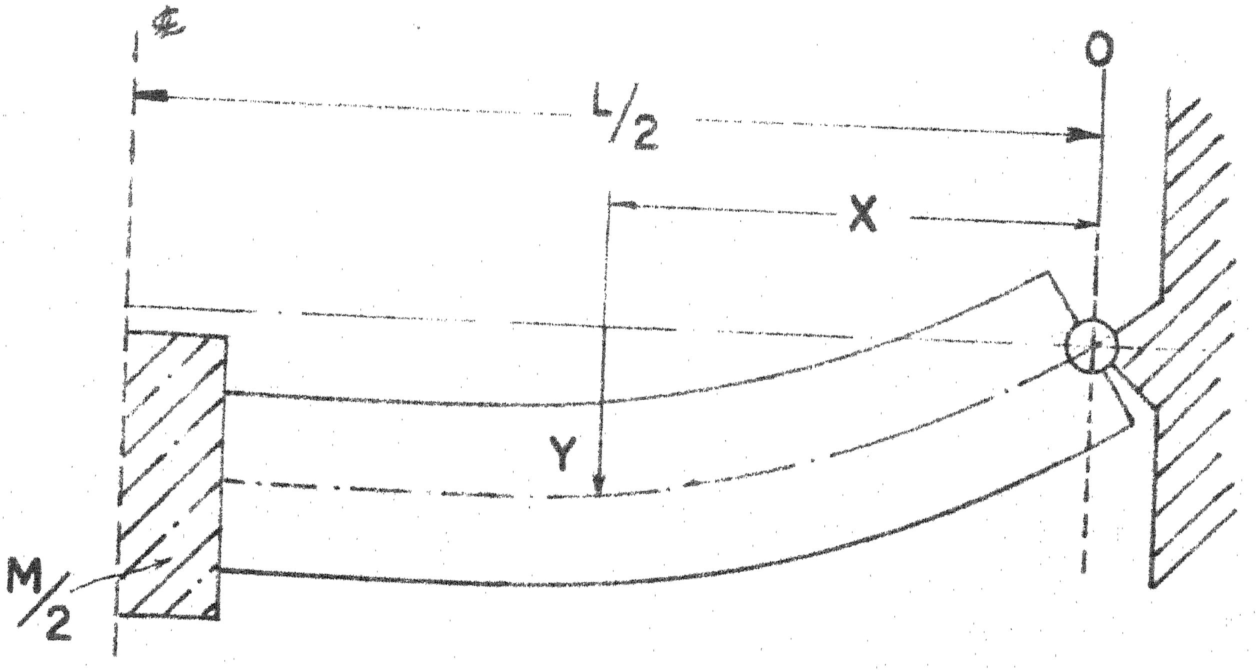

FIG. 2-1. SYMMETRIC MODE - SIMPLE SUPPORT

b) If M = 0, we get the frequency equation for supported-guided beam of span L/2 without mass. This equation is

$$\cos\frac{b\beta}{2} = 0 \tag{2.5}$$

c) When $\mathcal{M} \rightarrow \infty$ in equation (2.2), we get the frequency equation for supported - clamped beam of span L/2 without mass. This equation is

$$\lambda\xi \tanh\frac{b\alpha}{2} - \tan\frac{b\beta}{2} = 0 \tag{2.6}$$

This equation is same as Haung has obtained (20).

2) When $\left[(\gamma^2 - s^2)^2 + \frac{4}{b^2}\right]^{1/2} < (\gamma^2 + s^2)$,

frequency equation is

$$(1+\xi) = \frac{\mathcal{M}}{m}\cdot\frac{b}{2}\cdot\left\{\beta\cdot\tan\frac{b\beta}{2} + \alpha'\xi\cdot\tan\frac{b\alpha'}{2}\right\} \tag{2.7}$$

This equation is obtained by substituting $\alpha = i\alpha'$ in equation (2.2).

B. ASYMMETRIC MODES:

The beam shape for the first asymmetric mode is shown schematically in fig. 2-2. The boundary and continuity conditions are:

$$\begin{aligned} &\text{at } x = 0, \begin{cases} y = 0 \\ \partial\psi/\partial x = 0 \end{cases} \\ &\text{at } x = L/2, \begin{cases} y = 0 \\ M_o = -EI\partial\psi/\partial x = \frac{B}{2}\cdot\frac{\partial^2\psi}{\partial t^2} \end{cases} \end{aligned} \tag{2.9}$$

The frequency equations, obtained in the same manner as for the symmetric modes, are as follows:

1) When $\left[(\gamma^2 - s^2)^2 + \frac{4}{b^2}\right]^{1/2} > (\gamma^2 + s^2)$,

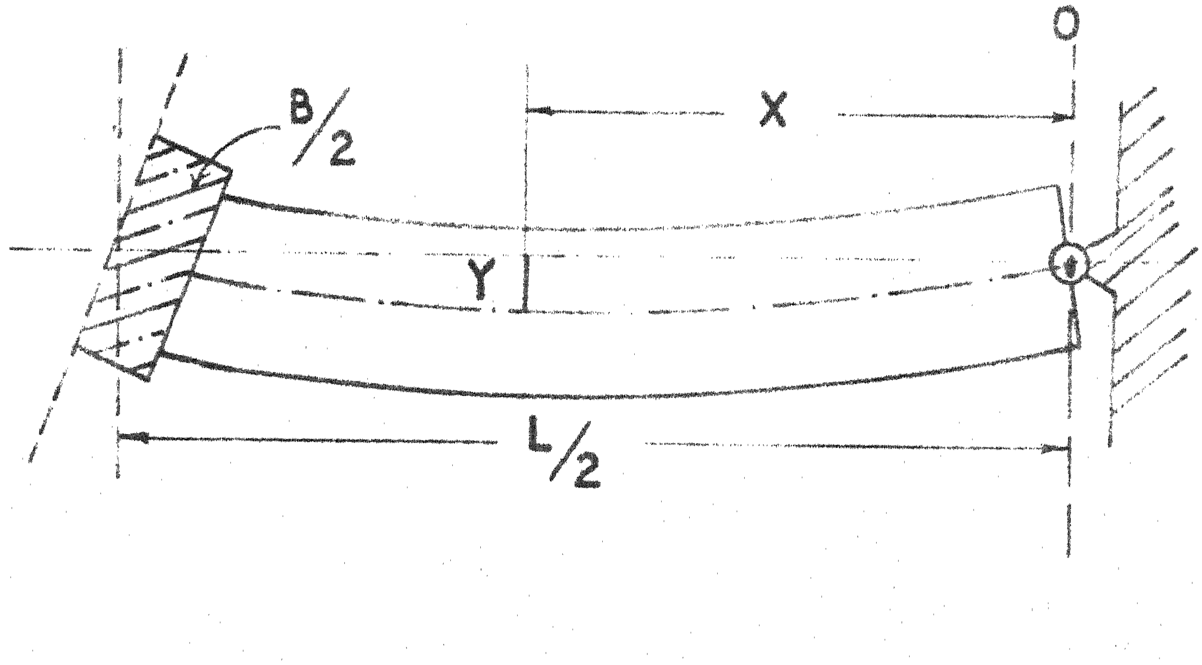

FIG. 2·2. ASYMMETRIC MODE SIMPLE SUPPORT

$$(1+\xi) = \frac{4B}{mL^2} \cdot \frac{b}{8} \left\{ \frac{1}{\alpha} \coth \frac{b\alpha}{2} - \frac{\xi}{\beta} \cot \frac{b\beta}{2} \right\} \qquad (2.10)$$

LIMITING CASES:

a) Ignoring the effect of shear deformation and rotatory inertia, equation (2.10) takes the following form

$$\lambda = \frac{4B}{mL^2} \cdot \left(\frac{\sqrt{b}}{2} \right)^3 \cdot \left[\coth \frac{\sqrt{b}}{2} - \cot \frac{\sqrt{b}}{2} \right] \qquad (2.11)$$

This equation is same as obtained by Baker (5).

b) For B = 0, the following equation is obtained

$$\sin \frac{b\beta}{2} = 0 \qquad (2.12)$$

This is the equation for supported - supported beam of span L/2 without mass. This equation is same as obtained by Haung (20).

c) For $B \rightarrow \infty$, equation (2.10) takes the following form

$$\lambda \xi \tanh \frac{b\alpha}{2} - \tan \frac{b\beta}{2} = 0 \qquad (2.6)$$

This is the frequency equation for supported - clamped beam of span L/2 without mass. This is same as obtained by Haung (20).

2) When $\left[(\gamma^2 - s^2)^2 + 4/b^2 \right]^{1/2} < (\gamma^2 + s^2)$,

by substituting $\alpha = i\alpha'$ in equation (2.10), we obtain the following equation

$$(1+\xi) = \frac{4B}{mL^2} \cdot \frac{b}{8} \cdot \left[-\frac{1}{\alpha'} \cdot \cot \frac{b\alpha'}{2} - \frac{\xi}{\beta} \cdot \cot \frac{b\beta}{2} \right] \qquad (2.14)$$

SECTION III

CLAMPED BEAM WITH CENTRAL MASS

A. SYMMETRIC MODES:

The boundary and continuity conditions can be written for the clamped beam as shown in fig. 3-1. These conditions are:

$$\text{at } x = 0, \quad \begin{cases} y = 0 \\ \psi = 0 \end{cases}$$

$$\text{at } x = L/2, \quad \begin{cases} \psi = 0 \\ Q = kAG\left(\frac{\partial y}{\partial x} - \psi\right) = -\frac{M}{2} \cdot \frac{\partial^2 y}{\partial t^2} \end{cases} \tag{3.1}$$

By substituting the equations (1.18) and (1.19) or equations (1.20) and (1.21) in (3.1), frequency equations can be found from the requirement that not all of the constants C_s can be zero. Frequency equations are:

1) When $\left[(r^2 - s^2)^2 + \frac{4}{b^2}\right]^{1/2} > (r^2 + s^2)$,

$$\frac{1}{\beta}(1+\xi)\left[\sinh\frac{b\alpha}{2}\cdot\cos\frac{b\beta}{2} + \lambda\xi\cdot\cosh\frac{b\alpha}{2}\cdot\sin\frac{b\beta}{2}\right] = \frac{M}{m}\cdot\frac{b}{2}\cdot\left[2\lambda\xi - 2\lambda\xi\cdot\cosh\frac{b\alpha}{2}\cdot\cos\frac{b\beta}{2} + (1-\lambda^2\xi^2)\cdot\sin\frac{b\beta}{2}\cdot\sinh\frac{b\alpha}{2}\right] \tag{3.2}$$

LIMITING CASES:

a) Ignoring the effect of shear deformation and rotatory inertia, equation (3.2) reduces to

$$\sinh\frac{\sqrt{b}}{2}\cdot\cos\frac{\sqrt{b}}{2} + \cosh\frac{\sqrt{b}}{2}\cdot\sin\frac{\sqrt{b}}{2} = \frac{M}{m}\cdot\frac{\sqrt{b}}{2}\left[1 - \cosh\frac{\sqrt{b}}{2}\cdot\cos\frac{\sqrt{b}}{2}\right] \tag{3.3}$$

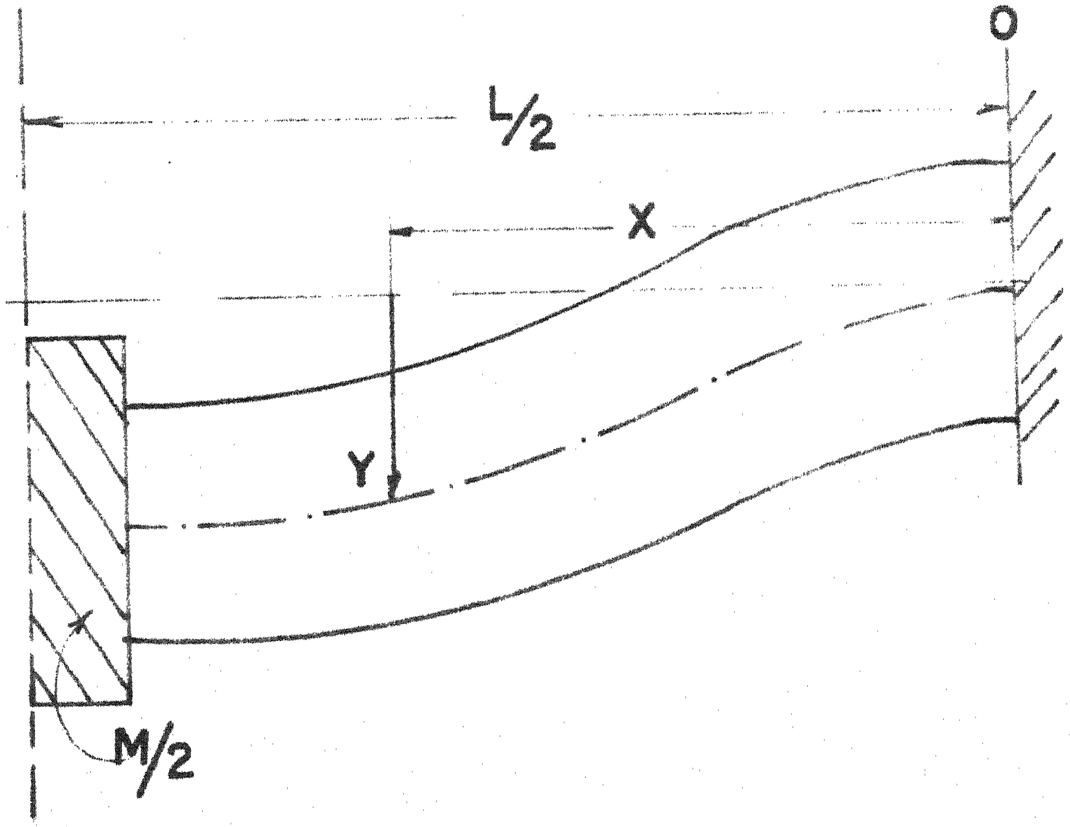

FIG. 3·1. SYMMETRIC MODE CLAMPED SUPPORTED

This equation is same as obtained by Baker (5).

b) If M = 0, we get the frequency equation for clamped - guided beam of span L/2 without mass. This equation is

$$\tanh\frac{b\alpha}{2} + \lambda\xi\tan\frac{b\beta}{2} = 0 \tag{3.4}$$

c) If $M \to \infty$, we get the frequency equation for clamped -clamped beam of span L/2 without mass. This equation is

$$2 - 2\cosh\frac{b\alpha}{2}\cdot\cos\frac{b\beta}{2} + \frac{b}{(1-b^2r^2s^2)^{1/2}}\cdot\left[b^2s^2(r^2-s^2)^2 + (3s^2-r^2)\right]\cdot\sinh\frac{b\alpha}{2}\cdot\sin\frac{b\beta}{2} = 0 \tag{3.5}$$

This is the same as obtained by Haung (20).

2) When $\left[(r^2-s^2)^2 + 4/b^2\right]^{1/2} < (r^2+s^2)$,

by substituting $\alpha = i\alpha'$ in equation (3.2), we obtained the following equation

$$\frac{1}{\beta}(1+\xi)\left[\sin\frac{b\alpha'}{2}\cdot\cos\frac{b\beta}{2} + \lambda'\xi\cdot\cos\frac{b\alpha'}{2}\cdot\sin\frac{b\beta}{2}\right] =$$
$$\frac{M}{m}\cdot\frac{b}{2}\cdot\left[2\lambda'\xi - 2\lambda'\xi\cdot\cos\frac{b\alpha'}{2}\cdot\cos\frac{b\beta}{2} + (1+\lambda'^2\xi^2)\cdot\sin\frac{b\alpha'}{2}\cdot\sin\frac{b\beta}{2}\right] \tag{3.6}$$

B. ASYMMETRIC MODES:

The beam shape for the first asymmetric mode is shown schematically in fig. 3-2. The boundary and continuity conditions are:

$$\text{at } x = 0, \qquad \begin{cases} y = 0 \\ \psi = 0 \end{cases}$$

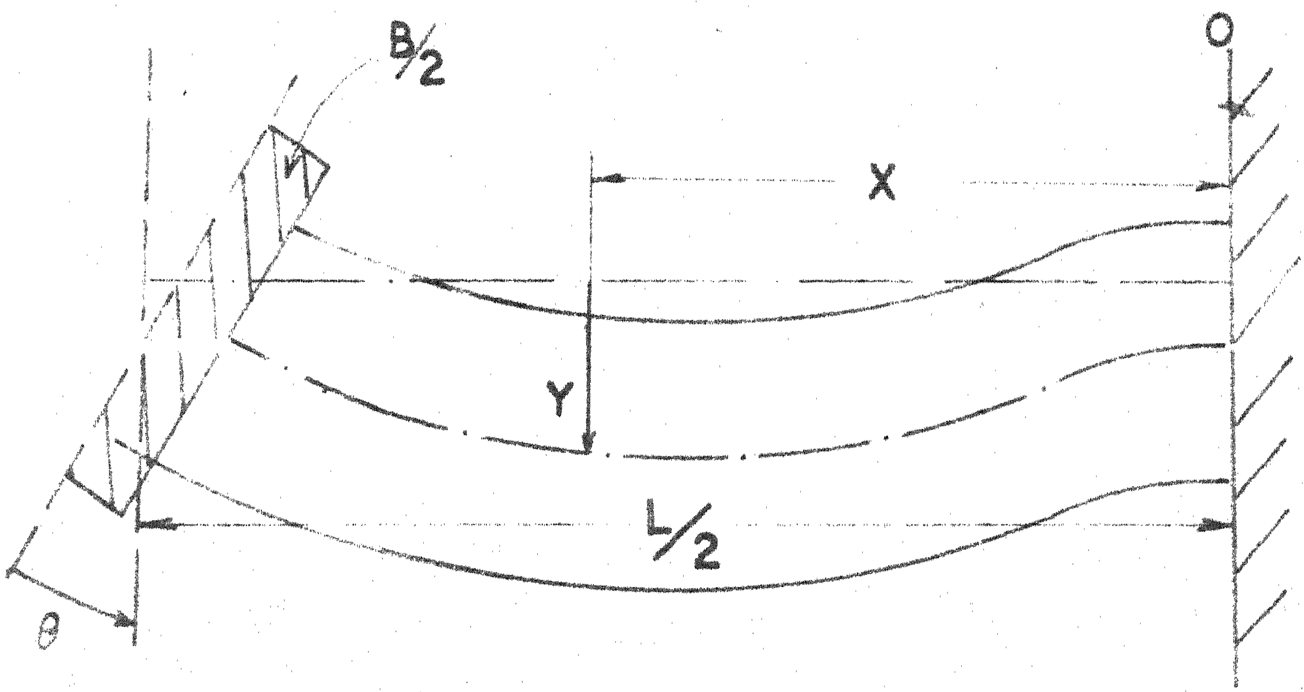

FIG.3-2. ASYMMETRIC MODE CLAMPED SUPPORTED

$$\text{at } x = L/2, \qquad \begin{cases} y = 0 \\ M_0 = -EI\frac{\partial\psi}{\partial x} = \frac{B}{2}\cdot\frac{\partial^2 y}{\partial t^2} \end{cases} \tag{3.7}$$

By substituting the equations (1.18) and (1.19) or equations (1.20) and (1.21) in (3.7), frequency equations can be found from the requirement that not all of the constants C_s can be zero. Frequency equations are:

1) When $\left[(\gamma^2 - s^2)^2 + \frac{4}{b^2}\right]^{1/2} > (\gamma^2 + s^2)$,

$$(\alpha + \lambda\zeta\beta)\cdot\left[\cos\frac{b\beta}{2}\cdot\sinh\frac{b\alpha}{2} - \frac{1}{\lambda\zeta}\cdot\sin\frac{b\beta}{2}\cdot\cosh\frac{b\alpha}{2}\right] = \frac{4B}{mL^2}\cdot\frac{b}{8}\cdot\left[2\cosh\frac{b\alpha}{2}\cdot\cos\frac{b\beta}{2} - 2 + \left(\lambda\zeta - \frac{1}{\lambda\zeta}\right)\cdot\sinh\frac{b\alpha}{2}\cdot\sin\frac{b\beta}{2}\right] \tag{3.8}$$

LIMITING CASES:

a) Ignoring the effect of shear deformation and rotatory inertia, we obtain the following equation from equation (3.8).

$$\cos\frac{\sqrt{b}}{2}\cdot\sinh\frac{\sqrt{b}}{2} - \sin\frac{\sqrt{b}}{2}\cdot\cosh\frac{\sqrt{b}}{2} = \frac{4B}{mL^2}\cdot\left(\frac{\sqrt{b}}{2}\right)^3\cdot\left[\cos\frac{\sqrt{b}}{2}\cosh\frac{\sqrt{b}}{2} - 1\right] \tag{3.9}$$

This equation is same as derived by Baker (5).

b) By substituting $B = 0$ in equation (3.9) we obtain the frequency equation for clamped supported beam of span L/2 without mass. This equation is

$$\lambda\zeta\tanh\frac{b\alpha}{2} - \tan\frac{b\beta}{2} = 0 \tag{3.10}$$

This equation same as derived by Haung (20).

c) When $B \to \infty$ in equation (3.9), the frequency equation is obtained for clamped -clamped beam of span L/2 without mass. This is same as equation (3.5).

2) When $\left[(\gamma^2-s^2)^2+4/b^2\right]^{1/2} < (\gamma^2+s^2)$,

$$(\alpha'+\lambda'\zeta\beta)\left[\sin\frac{b\alpha'}{2}\cdot\cos\frac{b\beta}{2}+\frac{1}{\lambda'\zeta}\cdot\cos\frac{b\alpha'}{2}\cdot\sin\frac{b\beta}{2}\right]=$$

$$\frac{4B}{mL^2}\cdot\frac{b}{8}\left[2-2\cos\frac{b\alpha'}{2}\cdot\cos\frac{b\beta}{2}+\left(\lambda'\zeta+\frac{1}{\lambda'\zeta}\right)\cdot\sin\frac{b\alpha'}{2}\cdot\sin\frac{b\beta}{2}\right] \quad (3.11)$$

SECTION IV

FREE BEAM WITH CENTRAL MASS

A. SYMMETRIC MODES:

The boundary and continuity conditions can be written for the free beam as shown in fig. 4-1 as follows:

$$\text{at } x = 0, \quad \begin{cases} \dfrac{\partial \psi}{\partial x} = 0 \\ \dfrac{\partial y}{\partial x} - \psi = 0 \end{cases}$$

$$\text{at } x = L/2, \quad \begin{cases} \psi = 0 \\ Q = kAG\left(\dfrac{\partial y}{\partial x} - \psi\right) = -\dfrac{M}{2}\cdot\dfrac{\partial^2 y}{\partial t^2} \end{cases} \tag{4.1}$$

By substituting the equations (1.18) and (1.19) or equations (1.20) and (1.21) in (4.1), frequency equations can be found from the requirement that not all of the constants C_s can be zero. Frequency equations are:

1) When $\left[(r^2-s^2)^2+\frac{4}{b^2}\right] > (r^2+s^2)$,

$$\xi/\beta(1+\xi)\sinh\frac{b\alpha}{2}\cdot\cos\frac{b\beta}{2}+\frac{\lambda}{\beta}(1+\zeta)\cosh\frac{b\alpha}{2}\cdot\sin\frac{b\beta}{2}=$$

$$\frac{M}{m}\cdot\frac{b}{2}\left\{\xi(1-\lambda^2)\sinh\frac{b\alpha}{2}\cdot\sin\frac{b\beta}{2}-\lambda(1+\xi^2)\cosh\frac{b\alpha}{2}\cdot\cos\frac{b\beta}{2}-2\lambda\xi\right\} \tag{4.2}$$

LIMITING CASES:

a) Ignoring the effect of shear deformation and rotatory inertia, we obtain the following equation from (4.2)

$$\sinh\frac{\sqrt{b}}{2}\cdot\cos\frac{\sqrt{b}}{2}+\cosh\frac{\sqrt{b}}{2}\cdot\sin\frac{\sqrt{b}}{2}=-\frac{M}{m}\cdot\frac{\sqrt{b}}{2}\left\{\cosh\frac{\sqrt{b}}{2}\cdot\cos\frac{\sqrt{b}}{2}+1\right\} \tag{4.3}$$

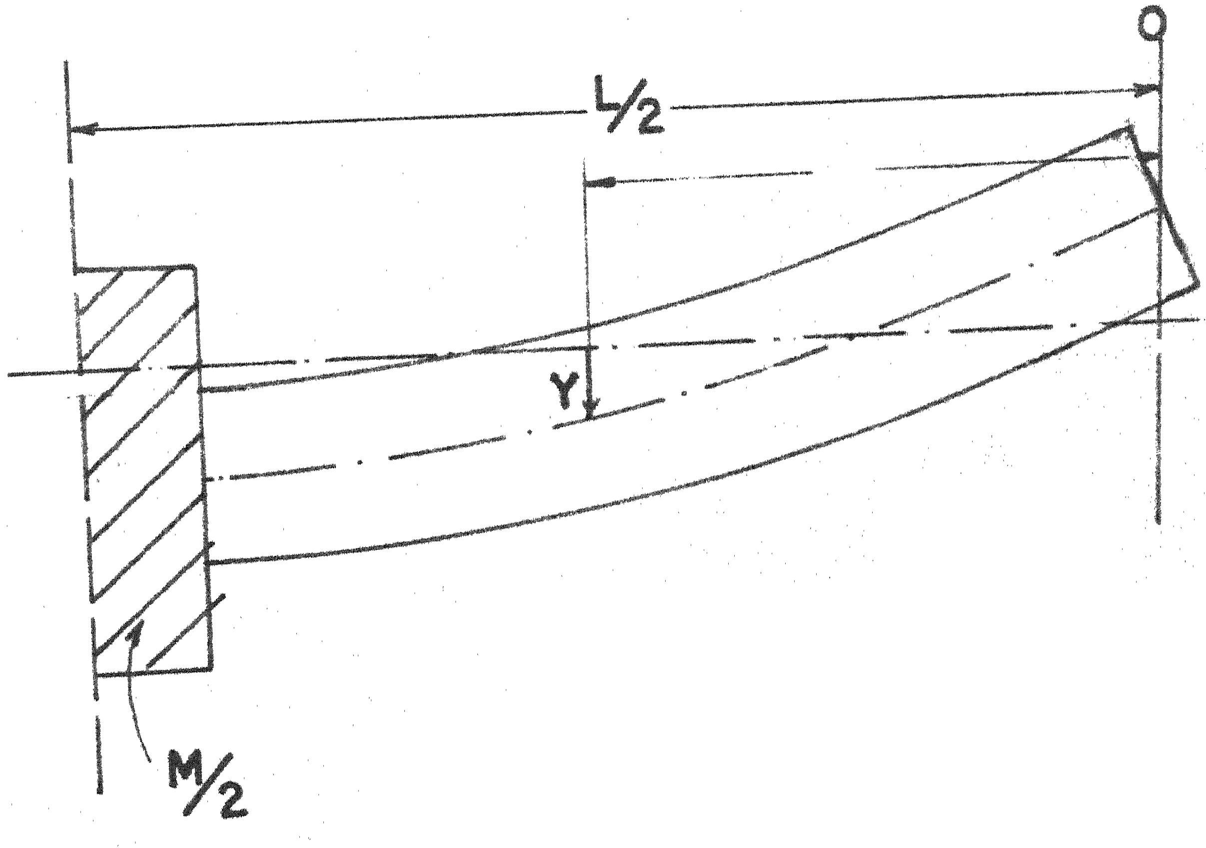

FIG.4·1. SYMMETRIC MODE-FREE BEAM

b) By substituting M = 0 in equation (4.2) we obtain the frequency equation for free - guided beam of span L/2 without mass. This equation is

$$\xi \tanh \frac{b\alpha}{2} + \lambda \tan \frac{b\beta}{2} = 0 \tag{4.4}$$

c) When $M \longrightarrow \infty$ in equation (4.2) we can easily obtain the frequency equation for free - claimped beam of span L/2 without mass. This is as follows:

$$2+\left[b^2(r^2-s^2)^2+2\right]\cosh\frac{b\alpha}{2}\cdot\cos\frac{b\beta}{2}-\frac{b(r^2+s^2)}{(1-b^2r^2s^2)^{1/2}}\sinh\frac{b\alpha}{2}\cdot\sin\frac{b\beta}{2}=0$$

This equation is obtained by Haung (20). (4.5)

2) When $\left[(r^2-s^2)^2+4/b^2\right]^{1/2} < (r^2+s^2)$,

$$\frac{1}{\beta}(1+\xi)\left\{\xi\sin\frac{b\alpha'}{2}\cdot\cos\frac{b\beta}{2}+\lambda'\cos\frac{b\alpha'}{2}\cdot\sin\frac{b\beta}{2}\right\}=$$

$$\frac{M}{m}\cdot\frac{b}{2}\left\{\xi(1+\lambda'^2)\cdot\sin\frac{b\alpha'}{2}\sin\frac{b\beta}{2}-\lambda'(1+\xi^2)\cdot\cos\frac{b\alpha'}{2}\cos\frac{b\beta}{2}-2\lambda'\xi\right\} \tag{4.6}$$

B. ASYMMETRIC MODES:

The boundary conditions can be written for this beam as shown in fig. 4-2 as follows:

$$\text{at } x=0,\quad \begin{cases} \dfrac{\partial\psi}{\partial x} = 0 \\ \dfrac{\partial y}{\partial x} - \psi = 0 \end{cases}$$

$$\text{at } x = L/2,\quad \begin{cases} y = 0 \\ M_o = -EI\dfrac{\partial\psi}{\partial x} = \dfrac{B}{2}\cdot\dfrac{\partial^2\psi}{\partial t^2} \end{cases} \tag{4.7}$$

The frequency equations are:

1) When $\left[(r^2-s^2)^2+\frac{4}{b^2}\right]^{1/2} > (r^2+s^2)$,

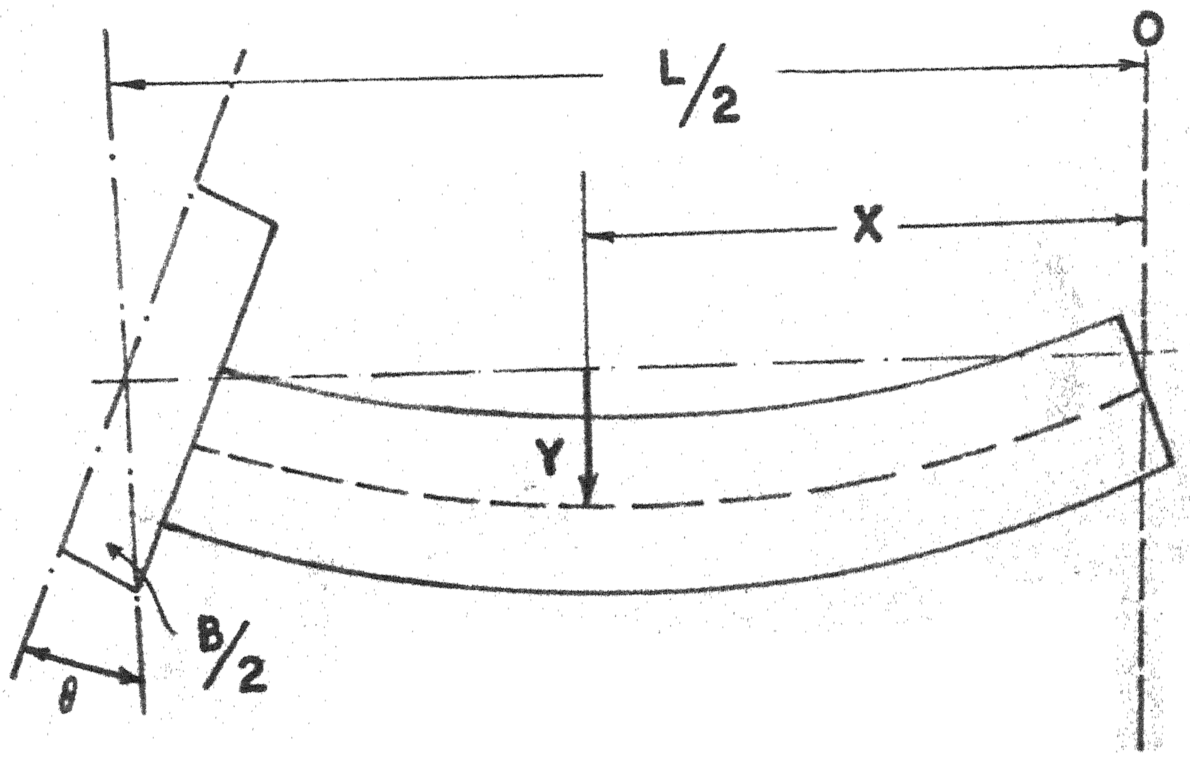

FIG. 4·2. ASYMMETRIC MODES-FREE BEAM

$$\alpha(1+\rho)\left\{\lambda \sinh\frac{b\alpha}{2}\cdot\cos\frac{b\beta}{2} - \rho\cosh\frac{b\alpha}{2}\cdot\sin\frac{b\beta}{2}\right\} =$$
$$\frac{4B}{mL^2}\cdot\frac{b}{8}\left\{2\lambda\rho + \lambda(1+\rho^2)\cdot\cosh\frac{b\alpha}{2}\cdot\cos\frac{b\beta}{2} - \rho(1-\lambda^2)\sinh\frac{b\alpha}{2}\cdot\sin\frac{b\beta}{2}\right\} \tag{4.8}$$

LIMITING CASES:

a) Ignoring the effect of shear deformation and rotatory inertia, we obtain the following equation from (4.8)

$$\sinh\frac{\sqrt{b}}{2}\cdot\cos\frac{\sqrt{b}}{2} - \cosh\frac{\sqrt{b}}{2}\cdot\sin\frac{\sqrt{b}}{2} = \frac{4B}{mL^2}\left\{\frac{\sqrt{b}}{2}\right\}^3\left[1+\cos\frac{\sqrt{b}}{2}\cdot\cosh\frac{\sqrt{b}}{2}\right] \tag{4.9}$$

b) Substituting B = 0 in equation (4.8), we obtain the frequency equation for free - supported beam of span L/2 without mass. This equation is

$$\lambda\tanh\frac{b\alpha}{2} - \rho\tan\frac{b\beta}{2} = 0 \tag{4.10}$$

This equation checks with equation derived by Haung (20).

c) When $B \to \infty$ in equation (4.8), we obtain the frequency equation (4.5) for free - clamped (cantilever) beam of span L/2 without mass.

2) When $\left[(r^2-s^2)^2+\frac{4}{b^2}\right]^{1/2} < (r^2+s^2)$,

$$\alpha'(1+\rho)\left[-\lambda'\sin\frac{b\alpha'}{2}\cdot\cos\frac{b\beta}{2} - \rho\cos\frac{b\alpha'}{2}\cdot\sin\frac{b\beta}{2}\right] =$$
$$\frac{4B}{mL^2}\cdot\frac{b}{8}\left\{2\lambda'\rho + \lambda'(1+\rho^2)\cos\frac{b\alpha'}{2}\cdot\cos\frac{b\beta}{2} - \rho(1+\lambda'^2)\cdot\sin\frac{b\alpha'}{2}\sin\frac{b\beta}{2}\right\} \tag{4.11}$$

SECTION V

BEAM WITH ENDS ELASTICALLY RESTRAINED AGAINST ROTATION AND A CENTRAL MASS

A. SYMMETRIC MODES:

The boundary and continuity conditions can be written for this beam as shown in fig. 5-1 as follows:

$$\text{at} \quad x = 0, \quad \begin{cases} y = 0 \\ EI\frac{\partial\psi}{\partial x} = j_o\,\psi \end{cases}$$

$$\text{at} \quad x = L/2, \quad \begin{cases} \psi = 0 \\ Q = kAG\left(\frac{\partial y}{\partial x} - \psi\right) = -\frac{M}{2}\cdot\frac{\partial^2 y}{\partial t^2} \end{cases} \tag{5.1}$$

By substituting the equations (1.18) and (1.19) or equations (1.20) and (1.21) in (5.1), frequency equation can be found from the requirement that not all of the constants C_s can be zero. Frequency equations are

1) When $\left[(r^2-s^2)^2+\frac{4}{b^2}\right]^{1/2} > (r^2+s^2)$,

$$\lambda(1+\xi)+\frac{M}{m}\cdot\frac{b\alpha}{2}\left\{\lambda\xi\tanh\frac{b\alpha}{2}-\tan\frac{b\beta}{2}\right\}+\frac{j_oL}{2EI}\cdot\frac{1}{b/2}\left\{\frac{1}{\beta}\left(\tanh\frac{b\alpha}{2}+\lambda\xi\tan\frac{b\beta}{2}\right)+\frac{M}{m}\cdot\frac{b}{2}\cdot\frac{\lambda\xi}{(1+\xi)}\left[2-2\,\mathrm{sech}\frac{b\alpha}{2}\cdot\sec\frac{b\beta}{2}-\left(\frac{1}{\lambda\xi}-\lambda\xi\right)\tanh\frac{b\alpha}{2}\cdot\tan\frac{b\beta}{2}\right]\right\}=0 \tag{5.2}$$

LIMITING CASES:

a) Ignoring the effect of shear deformation and rotatory inertia, we obtain the following equation from (5.2)

$$2+\frac{M}{m}\cdot\frac{\sqrt{b}}{2}\left\{\tanh\frac{\sqrt{b}}{2}-\tan\frac{\sqrt{b}}{2}\right\}+\frac{j_oL}{2EI}\cdot\left[\frac{M}{m}\cdot\left(1-\mathrm{sech}\frac{\sqrt{b}}{2}\cdot\sec\frac{\sqrt{b}}{2}\right)+\frac{1}{\sqrt{b}/2}\left(\tanh\frac{\sqrt{b}}{2}+\tan\frac{\sqrt{b}}{2}\right)\right]=0 \tag{5.3}$$

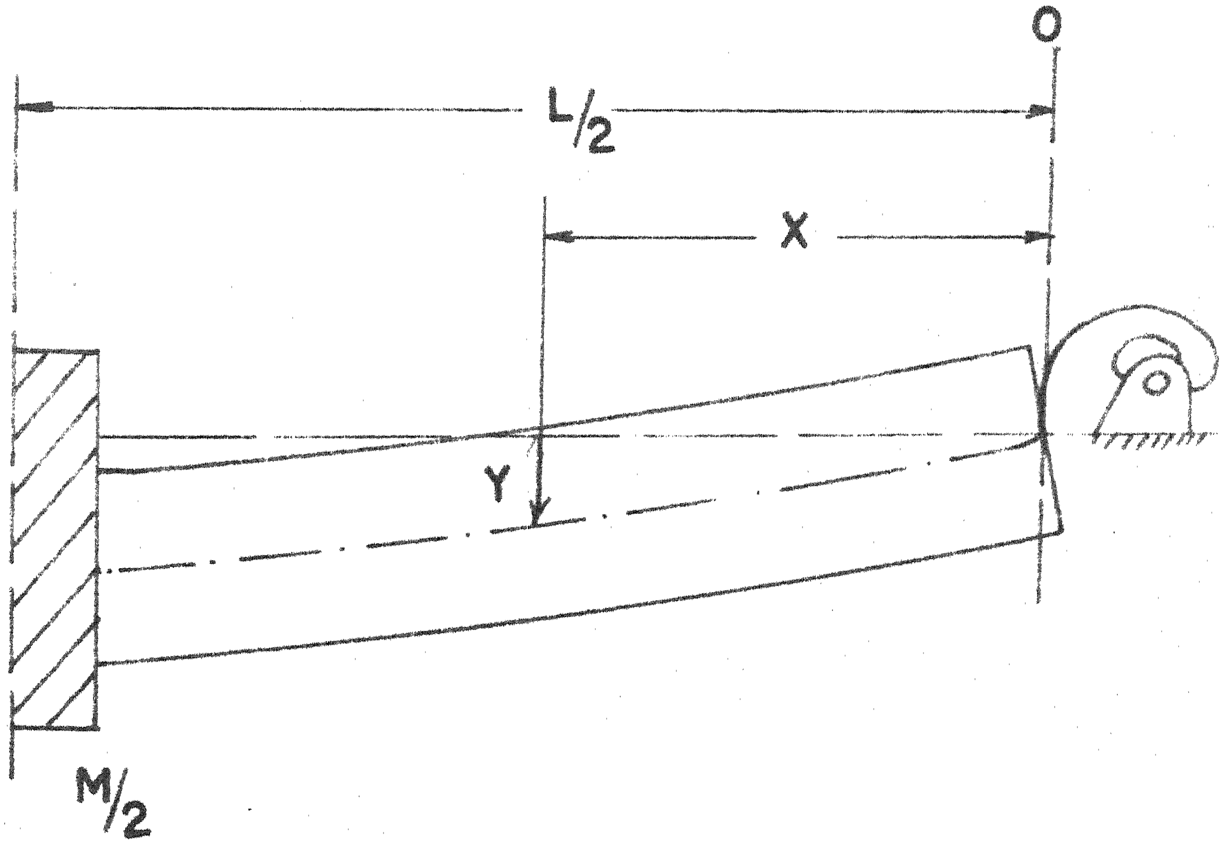

FIG.5·1. SYMMETRIC MODE-ELASTIC COIL SUPPORTED

This is the same equation which is obtained by Hess (6).

b) Substituting $j_o = 0$ in equation (5.2), we obtain the frequency equation for simple supported beam (2.2) which is already derived in section II.

c) As $j_o \rightarrow \infty$ in equation (5.2), we get the frequency equation for clamped - supported beam (3.2) which is given in section III.

d) Substituting $M = 0$ in equation (5.2), we obtain the following frequency equation for the end elastically restrained against rotation - guided beam of span L/2 without mass:

$$\lambda(1+\xi) + \frac{j_o L}{2EI} \cdot \frac{\beta}{b/2} \left\{ \tanh \frac{b\alpha}{2} + \lambda\xi \tan \frac{b\beta}{2} \right\} = 0 \tag{5.4}$$

e) When $M \rightarrow \infty$ in equation (5.2), we obtain the following frequency equation for elastically supported - clamped beam of span L/2 without mass

$$\frac{b\alpha}{2}\left\{\lambda\xi \tanh\frac{b\alpha}{2} - \tan\frac{b\beta}{2}\right\} + \frac{j_o L}{2EI} \cdot \left\{\frac{\lambda\xi}{(1+\xi)}\left[2 - 2\,\mathrm{sech}\frac{b\alpha}{2} \cdot \sec\frac{b\beta}{2} - \left(\frac{1}{\lambda\xi} - \lambda\xi\right)\tanh\frac{b\alpha}{2} \cdot \tan\frac{b\beta}{2}\right]\right\} = 0 \tag{5.5}$$

2) When $\left[(r^2 - s^2)^2 + \frac{4}{b^2}\right]^{1/2} < (r^2 + s^2)$,

$$\lambda'(1+\xi) - \frac{M}{m} \cdot \frac{b\alpha'}{2}\left[\lambda'\xi \tan\frac{b\alpha'}{2} + \tan\frac{b\beta}{2}\right] + \frac{j_o L}{2EI} \cdot \frac{1}{b/2}\left\{\frac{1}{\beta}\left(\tan\frac{b\alpha'}{2} + \lambda'\xi \tan\frac{b\beta}{2}\right) + \frac{M}{m} \cdot \frac{b}{2} \cdot \frac{\lambda'\xi}{(1+\xi)}\left[2 - 2\sec\frac{b\alpha'}{2} \cdot \sec\frac{b\beta}{2} - \left(\frac{1}{\lambda'\xi} + \lambda'\xi\right)\tan\frac{b\alpha'}{2} \cdot \tan\frac{b\beta}{2}\right]\right\} = 0 \tag{5.6}$$

B. ASYMMETRIC MODES:

The boundary and continuity conditions can be written for this beam as shown in fig. (5.2) as follows:

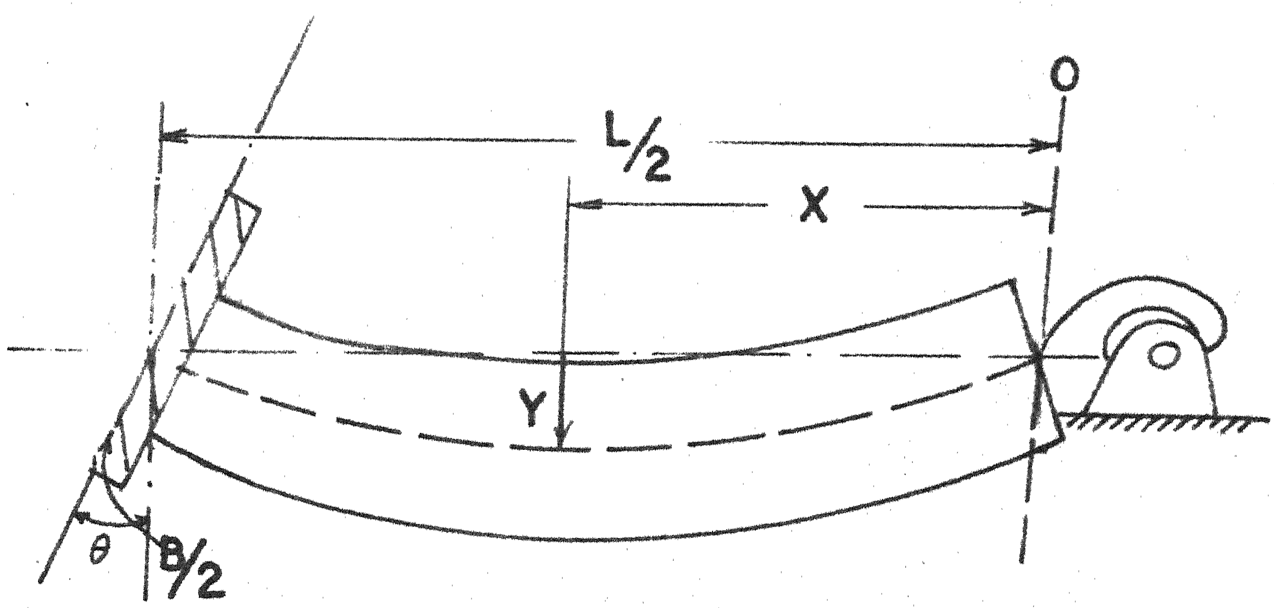

FIG. 5·2. ASYMMETRIC MODE-ELASTIC COIL SUPPORTED

$$\text{at } x = 0, \quad \begin{cases} y = 0 \\ EI\frac{\partial\psi}{\partial x} = j_o\psi \end{cases}$$

$$\text{at } x = L/2 \quad \begin{cases} y = 0 \\ M_o = -EI\frac{\partial\psi}{\partial x} = \frac{E}{2}\cdot\frac{\partial^2\psi}{\partial t^2} \end{cases} \qquad (5.7)$$

By substituting the equations (1.18) and (1.19) or equations (1.20) and (1.21) in (5.7), frequency equations can be found from the requirement that not all of the constants C_s can be zero. Frequency equations are:

1) When $\left[(\gamma^2-s^2)^2+\frac{4}{b^2}\right]^{1/2} > (\gamma^2+s^2)$,

$$\frac{j_oL}{2EI}\cdot\Big[2\alpha(1+\xi)\Big\{\cosh\frac{b\alpha}{2}\cdot\sin\frac{b\beta}{2}-\lambda\xi\sinh\frac{b\alpha}{2}\cdot\cos\frac{b\beta}{2}\Big\}-$$
$$\frac{4B}{mL^2}\cdot\frac{b}{4}\Big\{-2\lambda\xi\cosh\frac{b\alpha}{2}\cdot\cos\frac{b\beta}{2}+(1-\lambda^2\xi^2)\cdot\sinh\frac{b\alpha}{2}\cdot\sin\frac{b\beta}{2}+2\lambda\xi\Big\}\Big]$$
$$-\frac{4B}{mL^2}\cdot\frac{b}{8}\Big[b\alpha(1+\xi)\Big(\cosh\frac{b\alpha}{2}\cdot\sin\frac{b\beta}{2}-\lambda\xi\cdot\sinh\frac{b\alpha}{2}\cdot\cos\frac{b\beta}{2}\Big)\Big]+$$
$$b\alpha^2(1+\xi)^2\cdot\sinh\frac{b\alpha}{2}\cdot\sin\frac{b\beta}{2}=0 \qquad (5.8)$$

LIMITING CASES:

a) Ignoring the effect of shear deformation and rotatory inertia, we obtain the following equation from (5.8)

$$\sinh\frac{\sqrt{b}}{2}\cdot\sin\frac{\sqrt{b}}{2}\Big\{2+\Big[\Big(\frac{\sqrt{b}}{2}\Big)^3\cdot\frac{4B}{mL^2}-\frac{j_oL}{2EI}\cdot\Big(\frac{1}{\sqrt{b}/2}\Big)\Big]\cdot\Big[\cot\frac{\sqrt{b}}{2}-$$
$$\coth\frac{\sqrt{b}}{2}\Big]\Big\}+\frac{j_oL}{2EI}\cdot\frac{4B}{mL^2}\cdot\Big(\frac{\sqrt{b}}{2}\Big)^3\cdot\Big(\cos\frac{\sqrt{b}}{2}\cdot\cosh\frac{\sqrt{b}}{2}-1\Big)=0$$

$$(5.9)$$

This is the same equation as derived by Hess (6).

b) As $j_o \to o$ in equation (5.8), we obtain the frequency for simple supported beam (2.10) which is already

derived in section II.

c) As $j_o \to \infty$ in equation (5.8), we get the frequency equation for clamped - supported beam (3.8), which is given in section III.

d) Substituting $B = 0$ in equation (5.8), we obtain the following frequency equation for the end elastically restrained against rotation - supported beam of span L/2 without mass

$$\frac{j_o L}{\alpha EI}\left\{\cosh\frac{b\alpha}{2}\cdot\sin\frac{b\beta}{2} - 2\lambda\xi\sinh\frac{b\alpha}{2}\cdot\cos\frac{b\beta}{2}\right\} + b\alpha(1+\xi)\sinh\frac{b\alpha}{2}\cdot\sin\frac{b\beta}{2} = 0$$

(5.10)

e) As $B \to \infty$ in equation (5.8), we obtain the following frequency equation for the end elastically restrained against rotation - clamped beam of span L/2 without mass

$$+\frac{j_o L}{2EI}\left\{-2\lambda\xi\cosh\frac{b\alpha}{2}\cdot\cos\frac{b\beta}{2} + (1-\lambda^2\xi^2)\cdot\sinh\frac{b\alpha}{2}\cdot\sin\frac{b\beta}{2} + 2\lambda\xi\right\} + \frac{1}{\alpha}\cdot b\alpha(1+\xi)\left\{\cosh\frac{b\alpha}{2}\cdot\sin\frac{b\beta}{2} - \lambda\xi\sinh\frac{b\alpha}{2}\cdot\cos\frac{b\beta}{2}\right\} = 0$$

(5.11)

2) When $\left[(\gamma^2-s^2)^2+\frac{4}{b^2}\right]^{1/2} < (\gamma^2+s^2)$,

$$\frac{j_o L}{2EI}\cdot\left[2\alpha'(1+\xi)\left\{\cos\frac{b\alpha'}{2}\cdot\sin\frac{b\beta}{2} + 2\lambda'\xi\sin\frac{b\alpha'}{2}\cdot\cos\frac{b\beta}{2}\right\} - \frac{4B}{mL^2}\cdot\frac{b}{4}\left\{-2\lambda'\xi\cos\frac{b\alpha'}{2}\cdot\cos\frac{b\beta}{2} + (1+\lambda'^2\xi^2)\sin\frac{b\alpha'}{2}\cdot\sin\frac{b\beta}{2} + 2\lambda'\xi\right\}\right] - \frac{4B}{mL^2}\cdot\frac{b}{8}\left\{b\alpha'(1+\xi)\left(\cos\frac{b\alpha'}{2}\cdot\sin\frac{b\beta}{2} + \lambda'\xi\sin\frac{b\alpha'}{2}\cdot\cos\frac{b\beta}{2}\right\} - b\alpha'^2(1+\xi)^2\cdot\sin\frac{b\alpha'}{2}\cdot\sin\frac{b\beta}{2} = 0$$

(5.12)

SECTION VI

NUMERICAL SOLUTION OF FREQUENCY EQUATIONS

For a given beam with r and s known, the b_i $(i = 1,2,3, \ldots)$ can be found from the appropriate frequency equations and the corresponding p_i are then calculated by the equation

$$p_i^2 = b_i^2 \cdot \frac{EIg}{\gamma A L^4} \tag{6.1}$$

However, these frequency equations are highly transcendantal and not to be solved simply. This difficulty is overcome by the use of frequency charts which are obtained from the solution of these transcendantal frequency equations for various types of beams and various combinations of r and s.

In this paper the following data is assumed to solve these frequency equations numerically

$$k = 2/3$$

$$E/G = 8/3.$$

Hence, $E/kG = 4$ and $s = 2\ r$.

The range of r is taken from 0 to 0.10 with the increment of 0.02. The values of M/m and $4B/mL^2$ are taken from 0.0 to ∞.

Frequency equations are solved by trial and error method on IBM 1620 computer. Computer program is given on page 35. In this paper the solutions of the frequency equations are obtained in all the cases except for the ends, elastically restrained against rotation.

upto first two symmetric and two asymmetric modes. Same program can be used to obtain the higher modes.

The results are tabulated in the tables 1 to 12. These results are plotted graphically in figs. 6.1 to 6.12.

COMPUTER PROGRAMME

```
C     NUMERICAL SOLUTION OF FREQUENCY EQUATION BY TRIAL AND ERROR
C     METHOD
C     CLAMPED EDGES-ASYMMETRIC MODES
      DIMENSION F(2)
   98 FORMAT(3E14.7)
      J=1
    7 I=1
      DBR=0.1
C     AMR=MASS OR MOMENT OF INERTIA RATIO
C     BR=SQUARE ROOT OF B
C     R=MEASURE OF THE EFFECT OF ROTATORY INERTIA
C     S=MEASURE OF THE EFFECT OF SHEAR DEFORMATION
C     GIVING INITIAL ESTIMATE TO THE ROOT (BR)
      READ98,AMR,R,BR
      S=4
      RS=R*R
      SS=RS*S
C     FORMATION OF LEFT HAND SIDE OF THE EQUATION (F=0).BREAKING LEFT
C     HAND SIDE OF THE EQUATION INTO 2 OR 3 TERMS THAT IS F=F1+F2+--
   51 K=1
    4 B=BR*BR
      Q=SQRTF((RS-SS)**2+4./B**2)
      QN=RS+SS
      BT=SQRTF((QN+Q)/2.)
      IF(Q-QN)15,8,16
C     CASE 1
C     FORMATION OF F1,F2,ETC.
   16 AL=SQRTF((-QN+Q)/2.)
      X=B*BT/2.
      Y=B*AL/2.
      CH=(EXPF(Y)+EXPF(-Y))/2.
      SH=(EXPF(Y)-EXPF(-Y))/2.
      C=COSF(X)
      S=SINF(X)
      Q11=AL*AL+SS
      Q12=BT*BT-SS
      QR=Q12/Q11
      AMD=AL/BT
      F1=(AL+QR*AMD*BT)*(QR*AMD*C*SH-CH*S)
      F2=AMR*B/8.*(2.*QR*AMD*CH*C-2.*QR*AMD+(QR**2*AMD**2-1.)*SH*S)
      GO TO 17
```

```
C      CASE 2
C      FORMATION OF F1,F2,ETC.
   15  AL=SQRTF((QN-Q)/2.)
       X=B*BT/2.
       Y=B*AL/2.
       CH=COSF(Y)
       SH=SINF(Y)
       C=COSF(X)
       S=SINF(X)
       Q11=AL*AL-SS
       Q12=BT*BT-SS
       QR=Q12/Q11
       AMD=AL/BT
       F1=(AL-QR*AMD*BT)*(QR*AMD*C*SH-CH*S)
       F2=AMR*B/8.*(2.*QR*AMD-2.*QR*AMD*CH*C-(QR**2*AMD**2+1.)*SH*S)
   17  F(K)=F1-F2
       K=K+1
       BR=BR+DBR
       IF(K-2)4,4,11
   11  IF(F(1)*F(2))12,12,13
   13  F(1)=F(2)
       K=2
       GO TO 4
   12  BR=BR-2.*DBR
       I=I+1
       IF(I-4)71,71,82
C      TO FIND OUT THE ROOT TO THE NEXT DECIMAL
   71  DBR=DBR/10.
       GO TO 51
   82  PUNCH98,AMR,R,BR
       J=J+1
       IF(J-125)7,7,8
    8  STOP
       END
GO
```

TABLE 1

SIMPLY SUPPORTED - SYMMETRIC MODE 1

	M/m	0.0000	0.1000	1.000	10.0000	100.0000	∞
0.00	$\sqrt{b}$	3.1416	3.0013	2.3832	1.4627	0.8313	0.0000
	% Reduction	—	—	—	—	—	—
0.02	$\sqrt{b}$	3.1263	2.9871	2.3722	1.4558	0.8267	0.0000
	% Reduction	0.4800	0.4700	0.4600	0.4700	5.5200	—
0.04	$\sqrt{b}$	3.0831	2.9468	2.3409	1.4360	0.8151	0.0000
	% Reduction	1.8500	1.8200	1.7800	1.8200	1.9500	—
0.06	$\sqrt{b}$	3.0185	2.8864	2.2932	1.4059	0.7985	0.0000
	% Reduction	3.9200	3.8300	3.7700	3.8800	3.9500	—
0.08	$\sqrt{b}$	2.9401	2.8126	2.2344	1.3680	0.7766	0.0000
	% Reduction	6.4100	6.2900	6.2300	6.4700	6.5600	—
0.10	$\sqrt{b}$	2.8546	2.7317	2.1692	1.3268	0.7528	0.0000
	% Reduction	9.1300	8.9900	8.9600	9.2800	9.4300	—

TABLE 2

SIMPLY SUPPORTED - SYMMETRIC MODE 2

r_o \ M/m		0.0000	0.1000	1.0000	10.0000	100.0000	∞
0.00	$\sqrt{b}$	9.4247	9.0595	8.2394	7.9026	7.8582	7.8532
	% Reduction	—	—	—	—	—	—
0.02	$\sqrt{b}$	9.0555	8.7076	7.9000	7.5659	7.5220	7.5170
	% Reduction	3.9100	3.8800	4.0500	4.1800	4.2800	4.2900
0.04	$\sqrt{b}$	8.3006	7.9810	7.1959	6.8705	6.8280	6.8232
	% Reduction	11.9300	12.9800	12.6800	13.1000	13.1000	13.1200
0.06	$\sqrt{b}$	7.5481	7.2514	6.4922	6.1821	6.1422	6.1377
	% Reduction	19.9000	19.9500	21.2100	21.7600	21.8100	21.5800
0.08	$\sqrt{b}$	6.9054	6.6265	5.8975	5.6069	5.5700	5.5659
	% Reduction	26.6800	25.8200	28.0800	29.0000	29.1200	29.1800
0.10	$\sqrt{b}$	6.3736	6.1095	5.4128	5.1426	5.1089	5.1051
	% Reduction	32.3600	32.5600	34.1200	34.9000	34.9800	35.0000

TABLE 3

SIMPLY SUPPORTED - ASYMMETRIC MODE 1

r	$4B/mL^2$	0.0000	0.0100	0.1000	1.0000	10.0000	∞
0.00	$\sqrt{b}$	6.2832	5.9973	4.4627	2.6196	1.4795	0.0000
	% Reduction	–	–	–	–	–	–
0.02	$\sqrt{b}$	6.1362	5.8977	4.4417	2.6074	1.4724	0.0000
	% Reduction	1.8600	1.6600	0.4700	0.4700	0.4800	–
0.04	$\sqrt{b}$	5.8802	5.6887	4.3823	2.5724	1.4523	0.0000
	% Reduction	6.4200	5.1500	1.8000	1.8000	1.8400	–
0.06	$\sqrt{b}$	5.5337	5.4113	4.2913	2.5188	1.4217	0.0000
	% Reduction	11.9200	9.7800	3.8200	3.8200	3.9000	–
0.08	$\sqrt{b}$	5.1923	5.1167	4.1755	2.4518	1.3837	0.0000
	% Reduction	17.3200	14.6900	6.4500	6.4200	6.4800	–
0.10	$\sqrt{b}$	4.8805	4.8335	4.0385	2.3737	1.3414	0.0000
	% Reduction	22.3600	19.4500	9.5000	9.4000	11.1800	–

TABLE 4

SIMPLY SUPPORTED - ASYMMETRIC MODE 2

γ	$4B/mL^2$	0.0000	0.0100	0.1000	1.0000	10.0000	∞
0.00	$\sqrt{b}$	12.5660	10.2960	8.1970	7.8863	7.8562	7.8532
	% Reduction	—	—	—	—	—	—
0.02	$\sqrt{b}$	11.7604	9.9592	7.8452	7.5487	7.5202	7.5170
	% Reduction	4.0200	3.2700	4.2100	4.2700	4.2800	4.2800
0.04	$\sqrt{b}$	10.3846	9.2502	7.1245	6.8508	6.8250	6.8232
	% Reduction	17.3500	10.1600	13.0000	13.1200	13.1000	13.1000
0.06	$\sqrt{b}$	9.2072	8.4658	6.3967	6.1596	6.1398	6.1377
	% Reduction	26.6200	17.8000	21.9200	21.8000	21.7600	21.8200
0.08	$\sqrt{b}$	8.2901	7.5877	5.7709	5.5819	5.5675	5.5659
	% Reduction	34.0600	26.2400	29.5000	29.1600	29.0000	29.1600
0.10	$\sqrt{b}$	7.5735	6.5532	5.2491	5.1215	5.1061	5.1051
	% Reduction	39.7200	36.32000	35.9000	35.1000	35.0000	35.0000

TABLE 5

CLAMPED SUPPORTED - SYMMETRIC MODE 1

γ	M/m	0.0000	0.1000	1.0000	10.000	100.00	∞
0.00	$\sqrt{b}$	4.7300	4.4699	3.4377	2.0741	1.1760	0.0000
	% Reduction	–	–	–	–	–	–
0.02	$\sqrt{b}$	4.6357	4.3840	3.3749	2.0362	1.1545	0.0000
	% Reduction	1.9900	1.9200	1.8300	1.8300	1.8300	–
0.04	$\sqrt{b}$	4.4017	4.1696	3.2161	1.9401	1.0998	0.0000
	% Reduction	6.9500	6.7200	6.4400	6.4800	6.4800	–
0.06	$\sqrt{b}$	4.1151	3.9046	3.0171	1.8193	1.0312	0.0000
	% Reduction	12.960	12.650	12.250	12.300	12.300	–
0.08	$\sqrt{b}$	3.8331	3.6417	2.8176	1.6982	0.9625	0.0000
	% Reduction	18.920	18.520	18.060	18.150	18.150	–
0.10	$\sqrt{b}$	3.5784	3.4029	2.6350	1.5875	0.8997	0.0000
	% Reduction	24.320	23.820	23.380	23.500	23.540	–

TABLE 6

CLAMPED SUPPORTED - SYMMETRIC MODE 2

r	M/m	0.0000	0.1000	1.0000	10.0000	100.0000	∞
0.00	$\sqrt{b}$	10.9956	10.5887	9.7855	9.4908	9.4641	9.4600
	% Reduction	–	–	–	–	–	–
0.02	$\sqrt{b}$	10.2348	9.8733	9.1161	8.8416	8.8073	8.8034
	% Reduction	6.9200	6.5100	6.8300	6.9400	6.9400	6.9400
0.04	$\sqrt{b}$	8.9410	8.6409	7.954[illegible]	7.7013	7.6698	7.6663
	% Reduction	18.6400	17.6200	18.7300	18.9400	18.9400	18.9600
0.06	$\sqrt{b}$	7.8688	7.6073	6.9747	6.7428	6.7144	6.7112
	% Reduction	28.4200	27.1300	28.7300	28.9500	28.9200	29.0000
0.08	$\sqrt{b}$	7.0635	6.8251	6.2319	6.0182	5.9923	5.9894
	% Reduction	35.7200	34.2100	36.1800	36.6100	36.6200	36.6400
0.10	$\sqrt{b}$	6.4504	6.2270	5.6646	5.4661	5.4423	5.4397
	% Reduction	41.2000	40.0600	42.1000	42.5000	42.4600	42.5000

TABLE 7

CLAMPED SUPPORTED - ASYMMETRIC MODE #1

r	$4B/mL^2$	0.0000	0.0100	0.1000	1.0000	10.0000	∞
0.00	$\sqrt{b}$	7.8532	7.2122	4.9095	2.8216	1.5901	0.0000
	% Reduction	–	–	–	–	–	–
0.02	$\sqrt{b}$	7.5170	7.0065	4.8398	2.7831	1.5684	0.0000
	% Reduction	4.2800	2.8500	1.4200	1.3600	1.3600	–
0.04	$\sqrt{b}$	6.8232	6.5164	4.6660	2.6879	1.5149	0.0000
	% Reduction	13.1200	9.6600	4.9500	4.7400	4.7200	–
0.06	$\sqrt{b}$	6.1377	5.9577	4.4524	2.5740	1.4510	0.0000
	% Reduction	21.8200	17.3800	9.3200	8.7500	8.7500	–
0.08	$\sqrt{b}$	5.5659	5.4518	4.2418	2.4671	1.3914	0.0000
	% Reduction	29.1000	24.4200	13.6100	12.5500	12.5000	–
0.10	$\sqrt{b}$	5.1051	5.0262	4.0506	2.3766	1.3414	0.0000
	% Reduction	35.0000	34.0000	17.5000	15.7600	15.6200	–

TABLE 8

CLAMPED SUPPORTED - ASYMMETRIC MODE 2

γ \ $4B/mL^2$		0.0000	0.0100	0.1000	1.0000	10.0000	∞
0.00	$\sqrt{b}$	14.1371	11.2575	9.6543	9.4790	9.4619	9.4600
	% Reduction	–	–	–	–	–	–
0.02	$\sqrt{b}$	12.7586	10.6354	8.9925	8.8216	8.8052	8.8034
	% Reduction	9.7600	5.5400	6.8600	6.9600	6.9600	6.9600
0.04	$\sqrt{b}$	10.7990	9.5784	7.8463	7.6828	7.6680	7.6663
	% Reduction	24.0000	14.9100	18.7100	18.9500	18.9500	18.9500
0.06	$\sqrt{b}$	9.3620	8.6518	6.8826	6.7261	6.7126	6.7112
	% Reduction	33.7200	23.1000	28.9800	29.0100	29.0100	29.0100
0.08	$\sqrt{b}$	8.3291	7.8055	6.1529	6.0030	5.9907	5.9894
	% Reduction	41.0000	30.6200	36.3100	36.6200	36.6200	36.6200
0.10	$\sqrt{b}$	7.8628	6.9485	5.5954	5.4522	5.4409	5.4397
	% Reduction	44.4000	38.2000	42.0800	42.5000	42.5000	42.5000

TABLE 9

FREE SUPPORTED - SYMMETRIC MODE 1

r_o \ M/m		0.0000	0.1000	1.0000	10.000	100.000	∞
0.00	$\sqrt{b}$	4.7300	4.5823	4.1079	3.8043	3.7559	3.7502
	% Reduction	—	—	—	—	—	—
0.02	$\sqrt{b}$	4.6843	4.5402	4.0715	3.7700	3.7219	3.7162
	% Reduction	0.9600	0.9200	0.8700	0.9000	0.9100	0.0100
0.04	$\sqrt{b}$	4.5649	4.4279	3.9733	3.6771	3.6297	3.6241
	% Reduction	3.4900	3.3600	3.2200	3.3400	3.3700	3.3700
0.06	$\sqrt{b}$	4.4025	4.2746	3.8371	3.5473	3.5010	3.4955
	% Reduction	6.9200	6.7000	6.4800	6.7600	6.8000	6.8000
0.08	$\sqrt{b}$	4.2254	4.1063	3.6850	3.4016	3.3563	3.3509
	% Reduction	10.6500	10.4000	10.0100	11.2200	10.6500	10.6500
0.10	$\sqrt{b}$	4.0501	3.9387	3.5314	3.2540	3.2096	3.2044
	% Reduction	14.3800	14.0300	13.7600	14.4600	14.5500	14.5500

TABLE 10

FREE SUPPORTED - SYMMETRIC MODE 2

r	M/m	0.0000	0.1000	1.0000	10.0000	100.0000	∞
0.0	$\sqrt{b}$	10.9956	10.5774	9.7372	9.4312	9.3825	9.3810
	% Reduction	-	-	-	-	-	-
0.02	$\sqrt{b}$	10.4500	10.0520	9.2153	8.9035	8.8699	8.8655
	% Reduction	4.9500	4.9800	5.3600	5.5400	5.5500	5.5000
0.04	$\sqrt{b}$	9.4210	9.0476	8.2144	7.9127	7.8753	7.8710
	% Reduction	14.6000	14.4600	15.6500	16.1000	16.1400	16.1000
0.06	$\sqrt{b}$	8.4500	8.0946	7.2802	6.9963	6.9617	6.9578
	% Reduction	23.1500	23.4600	25.2000	25.8000	25.8200	25.8200
0.08	$\sqrt{b}$	7.6251	7.2906	6.5194	6.2620	6.2313	6.2278
	% Reduction	30.6000	31.1000	33.0000	33.6000	33.6100	33.6000
0.10	$\sqrt{b}$	6.9163	6.6149	5.9100	5.6824	5.6557	5.6527
	% Reduction	37.0800	37.5000	39.4000	39.7600	39.7000	39.8000

TABLE 11

FREE SUPPORTED - ASYMMETRIC MODE 1

γ \ $4B/mL^2$		0.0000	0.0100	0.1000	1.0000	10.000	∞
0.00	$\sqrt{b}$	7.8532	7.2810	5.2778	4.0200	3.7801	3.7502
	% Reduction	-	-	-	-	-	-
0.02	$\sqrt{b}$	7.6421	7.1680	5.2387	3.9852	3.7461	3.7162
	% Reduction	2.6800	1.5500	0.7400	0.8600	0.9000	0.9100
0.04	$\sqrt{b}$	7.1640	6.8756	5.1339	3.8912	3.6538	3.6241
	% Reduction	8.7700	5.5600	2.7300	3.2000	3.3400	3.3600
0.06	$\sqrt{b}$	6.6288	6.4906	4.9878	3.7599	3.5249	3.4955
	% Reduction	15.6000	10.8500	5.5000	6.4600	6.7500	6.8000
0.08	$\sqrt{b}$	6.1195	6.0730	4.8179	3.6121	3.3801	3.3509
	% Reduction	22.0600	16.5800	8.7200	10.1400	10.5800	10.6500
0.10	$\sqrt{b}$	5.6427	5.6384	4.6270	3.4616	3.2332	3.2044
	% Reduction	28.1800	22.5000	12.3300	13.8600	14.4600	14.6000

TABLE 12

FREE SUPPORTED - ASYMMETRIC MODE 2

γ \ $4B/mL^2$		0.0000	0.01000	0.1000	1.0000	10.0000	∞
0.000	$\sqrt{b}$	14.1371	11.2310	9.5875	9.4075	9.3901	9.3810
	% Reduction	-	-	-	-	-	-
0.02	$\sqrt{b}$	13.0669	10.7274	9.0531	8.8834	8.8673	8.8655
	% Reduction	7.5700	4.4800	5.6000	5.5700	5.5700	5.5000
0.04	$\sqrt{b}$	11.3622	9.7313	8.0229	7.8834	7.8724	7.8710
	% Reduction	19.4500	13.3600	16.3200	16.2000	16.1800	16.0800
0.06	$\sqrt{b}$	9.9191	8.6634	7.0563	6.9661	6.9586	6.9578
	% Reduction	29.8100	28.6100	26.4000	25.9400	25.9200	25.8300
0.08	$\sqrt{b}$	8.5782	7.5918	6.2705	6.2311	6.2281	6.2278
	% Reduction	39.3600	32.4200	34.5400	33.7000	33.6000	33.6000
0.10	$\sqrt{b}$	7.2515	6.6510	5.6576	5.6531	5.6528	5.6527
	% Reduction	48.7000	40.7000	41.0000	39.5000	39.7600	39.7600

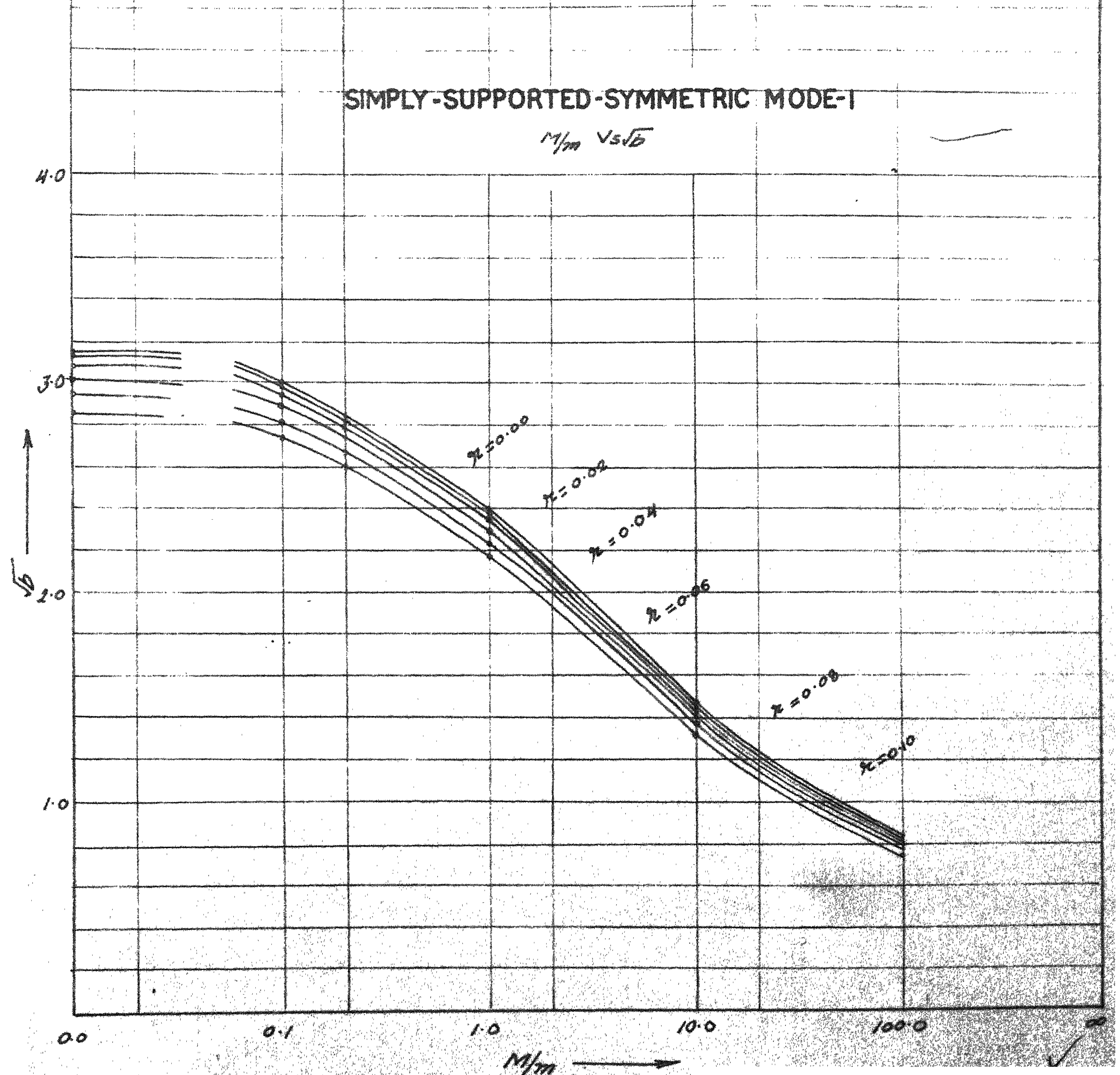
SIMPLY-SUPPORTED-SYMMETRIC MODE-I
M/m Vs √b
4.0
3.0
2.0
1.0
√b
r = 0.00
r = 0.02
r = 0.04
r = 0.06
r = 0.08
r = 0.10
0.0
0.1
1.0
10.0
100.0
∞
M/m

FIG. 6.1

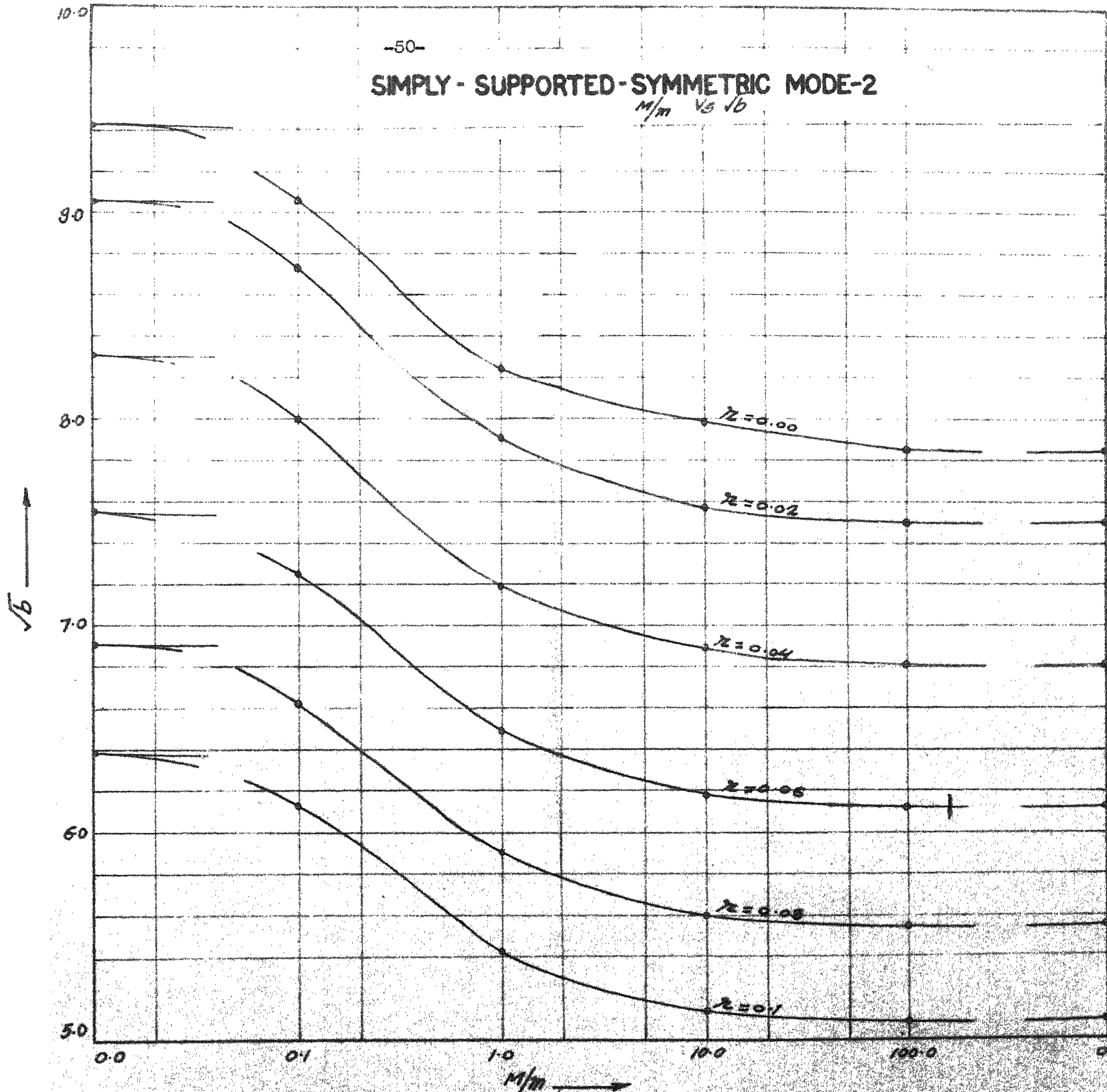
SIMPLY - SUPPORTED - SYMMETRIC MODE-2
M/m Vs √b
10.0
9.0
8.0
7.0
6.0
5.0
√b
0.0
0.1
1.0
10.0
100.0
∞
M/m
n = 0.00
n = 0.02
n = 0.04
n = 0.06
n = 0.08
n = 0.1

FIG. 6.2

SIMPLY SUPPORTED-ASYMMETRIC MODE-I

4B/mL2 VS/b

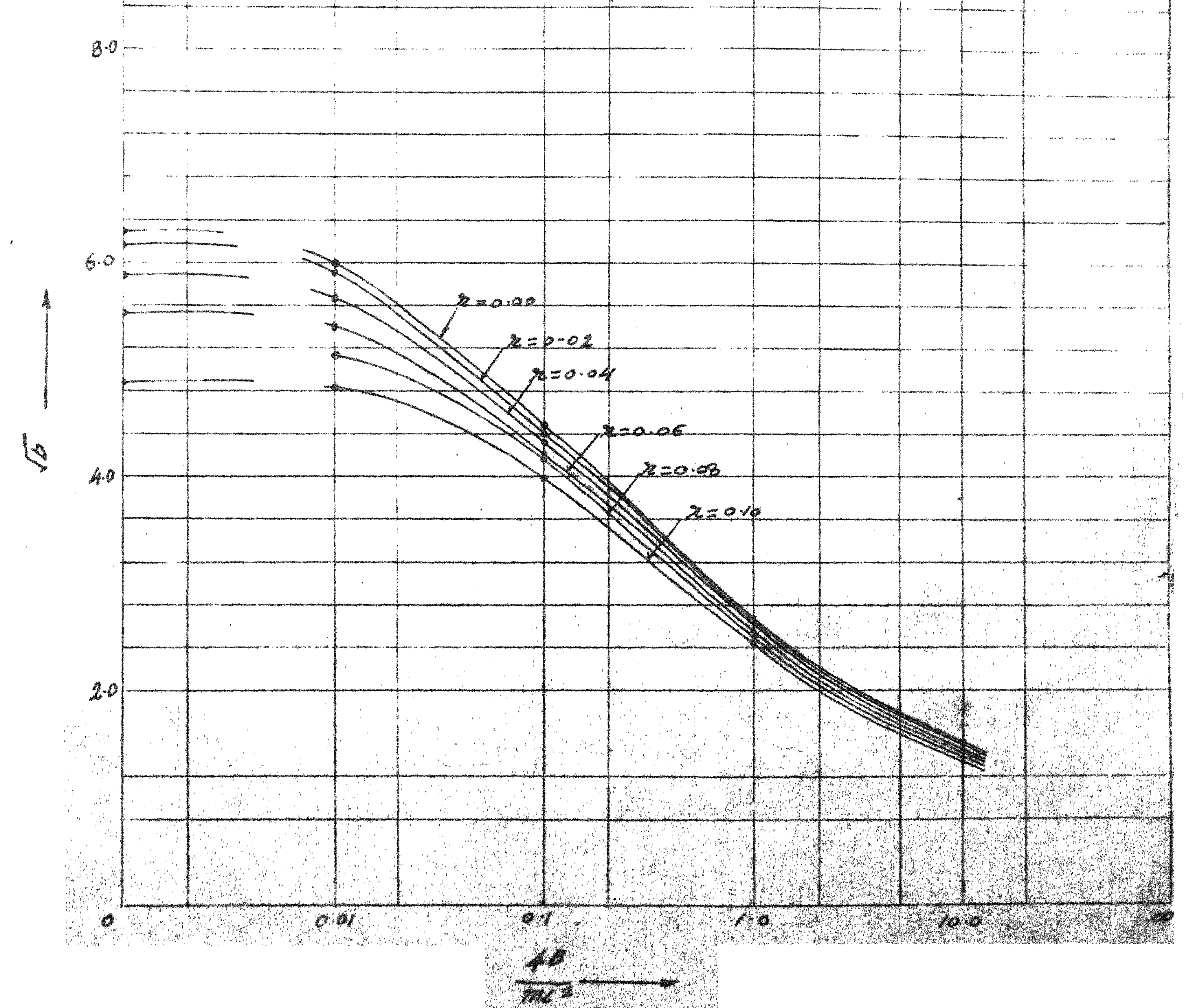

FIG. 6.3

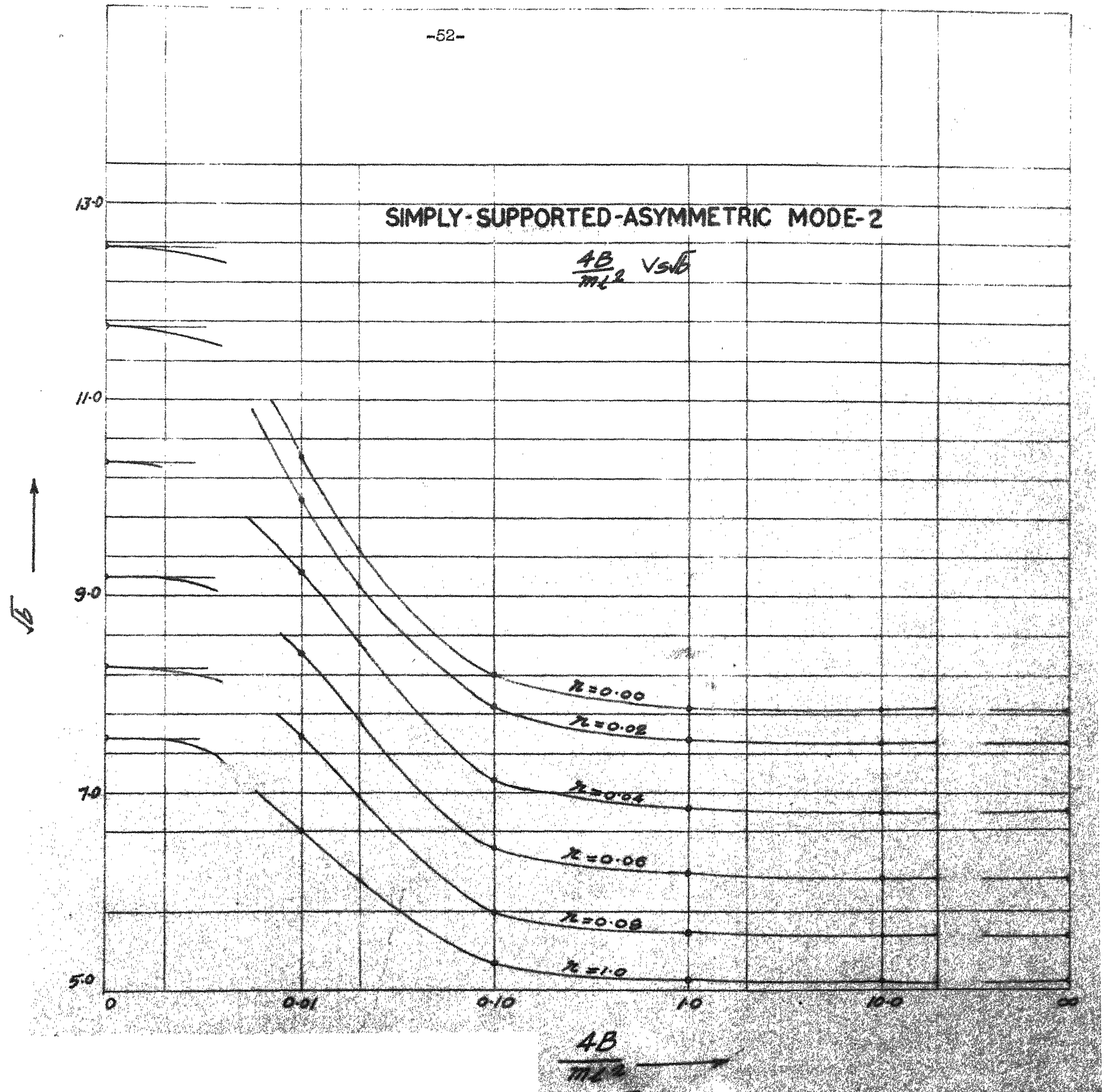
SIMPLY-SUPPORTED-ASYMMETRIC MODE-2
$\frac{4B}{ml^2}$ Vs $\sqrt{b}$
$\sqrt{b}$
13·0
11·0
9·0
7·0
5·0
0
0·01
0·10
1·0
10·0
∞
$\frac{4B}{ml^2}$

FIG. 6

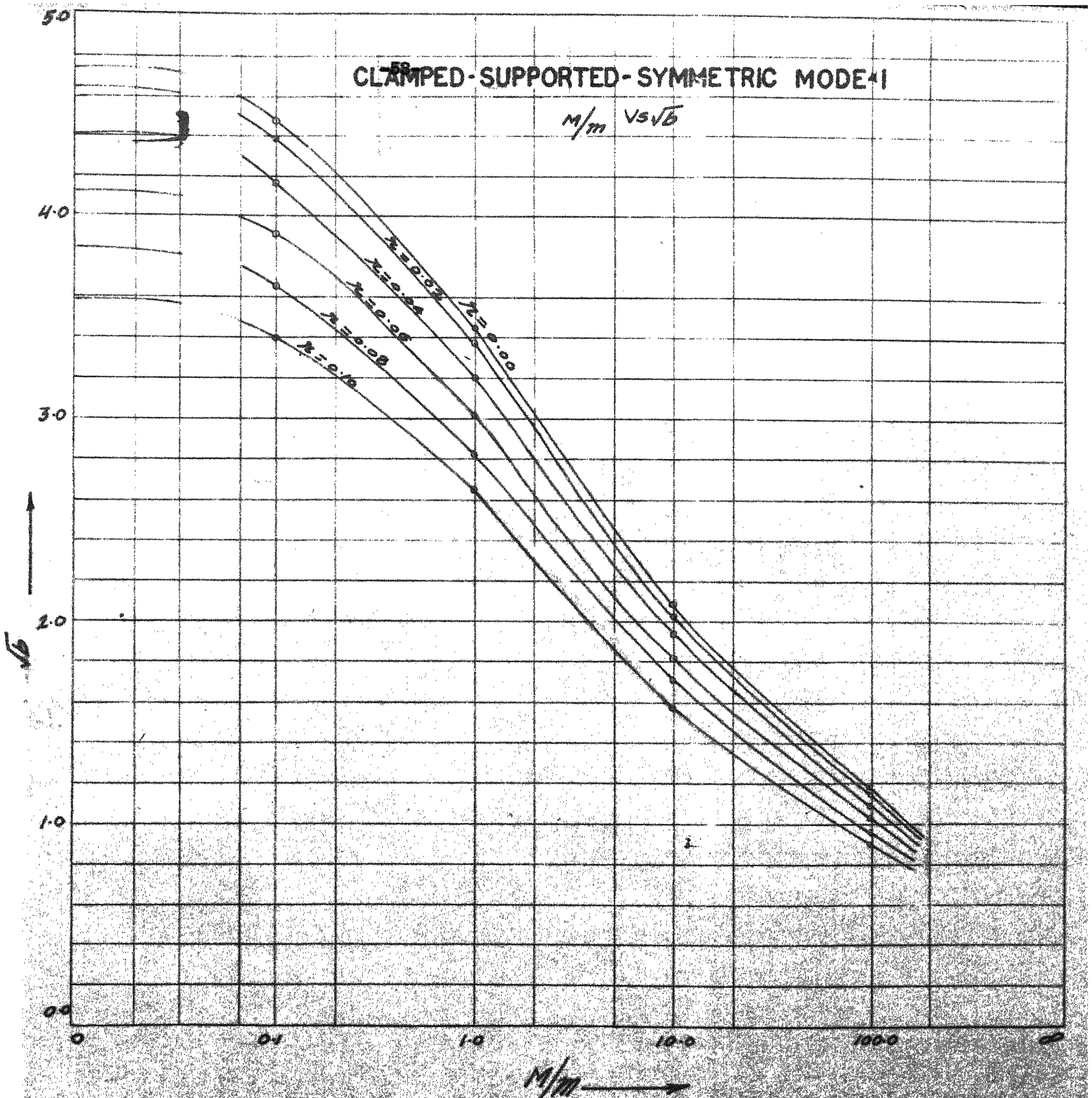
CLAMPED-SUPPORTED-SYMMETRIC MODE-I
M/m VS √b
5.0
4.0
3.0
2.0
1.0
0.0
√b
0
0.1
1.0
10.0
100.0
M/m

FIG. 6.5

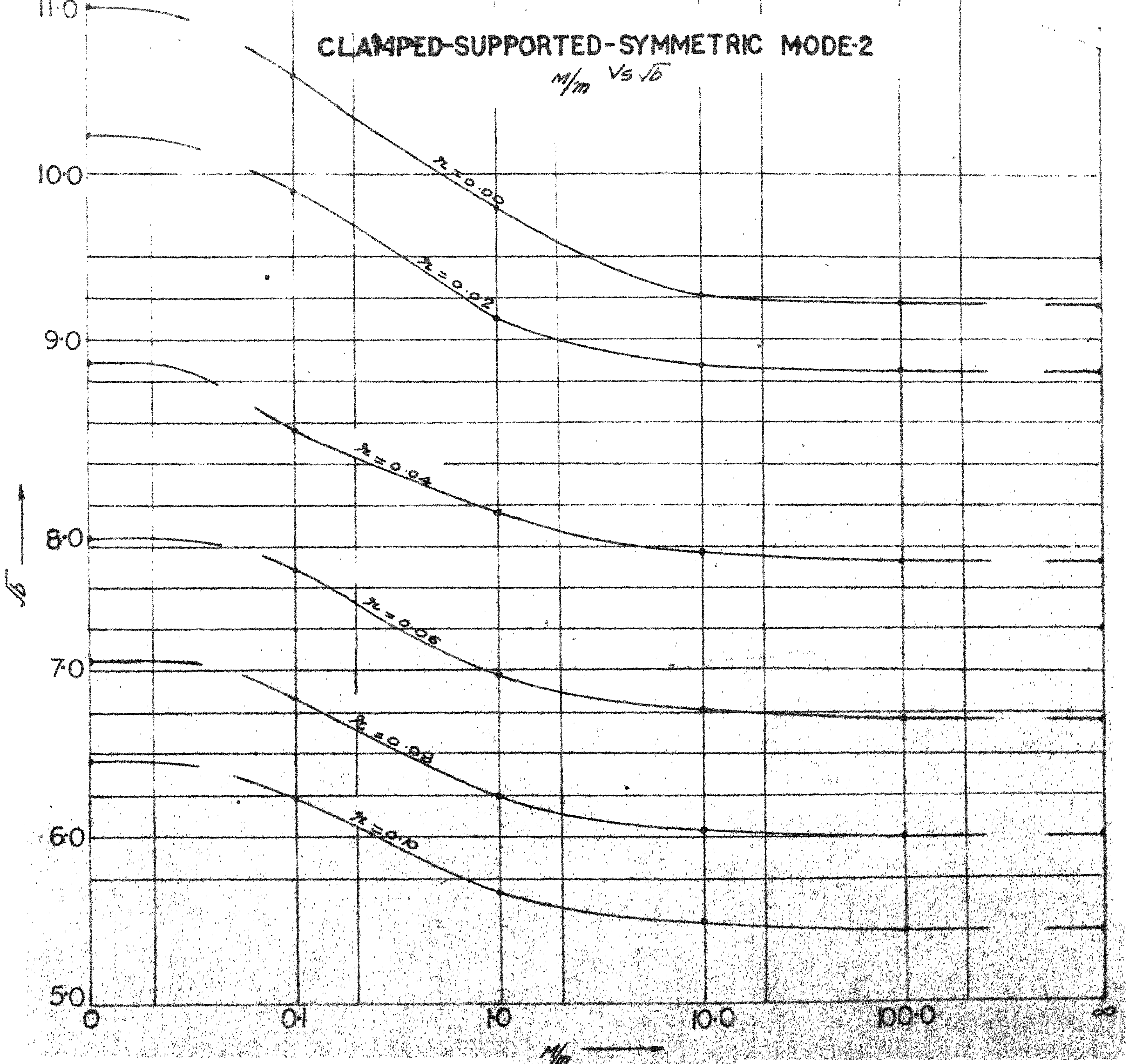
CLAMPED-SUPPORTED-SYMMETRIC MODE-2
M/m Vs √b
r = 0.00
r = 0.02
r = 0.04
r = 0.06
r = 0.08
r = 0.10
11.0
10.0
9.0
8.0
7.0
6.0
5.0
√b
0
0.1
1.0
10.0
100.0
∞
M/m

FIG. 6.6

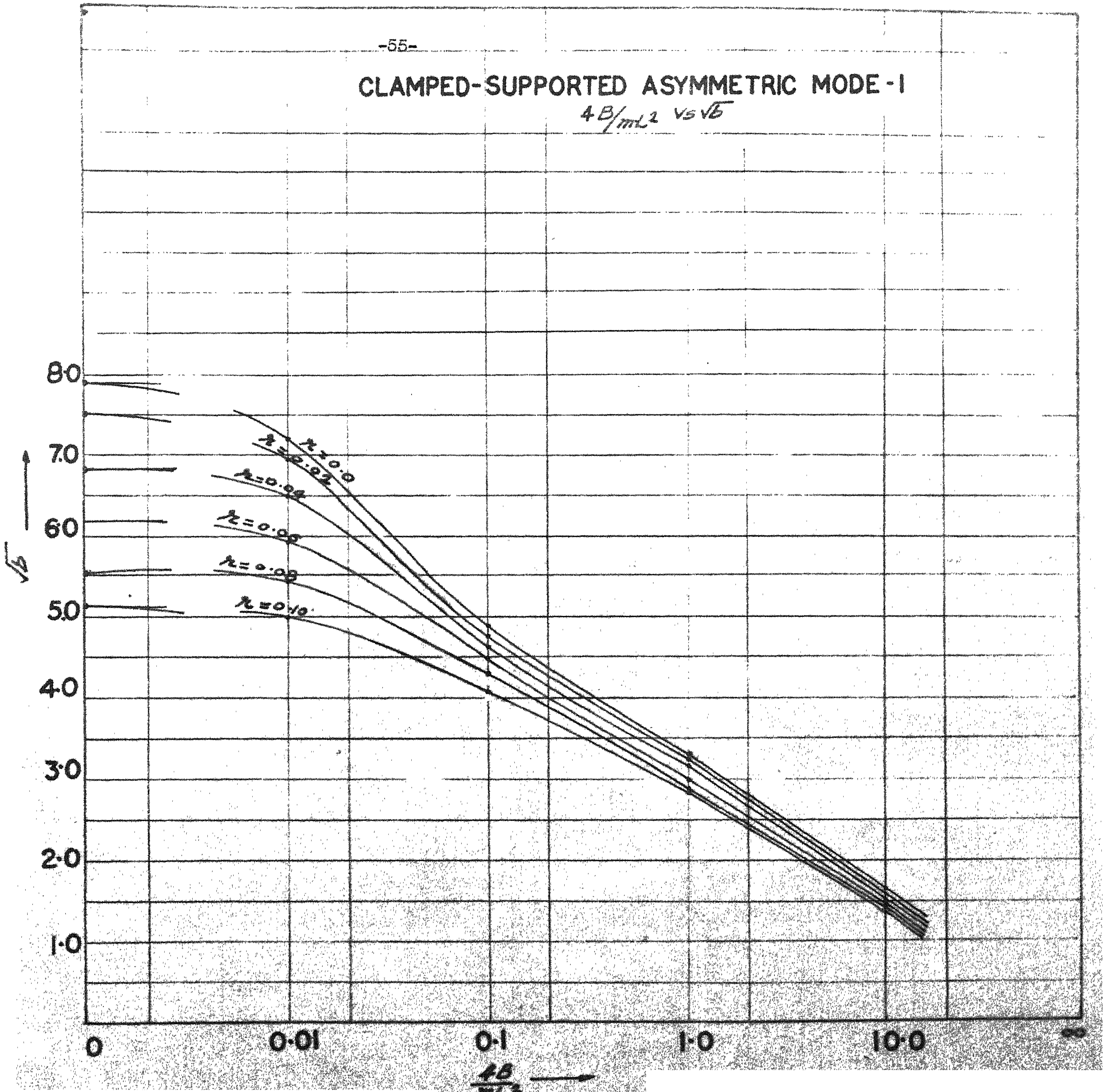
CLAMPED-SUPPORTED ASYMMETRIC MODE-I
4B/mL² vs √b
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
√b
0
0.01
0.1
1.0
10.0
∞
4B/mL²
r=0.0
r=0.02
r=0.04
r=0.06
r=0.08
r=0.10

FIG. 6.7

CLAMPED-SUPPORTED ASYMMETRIC MODE-2

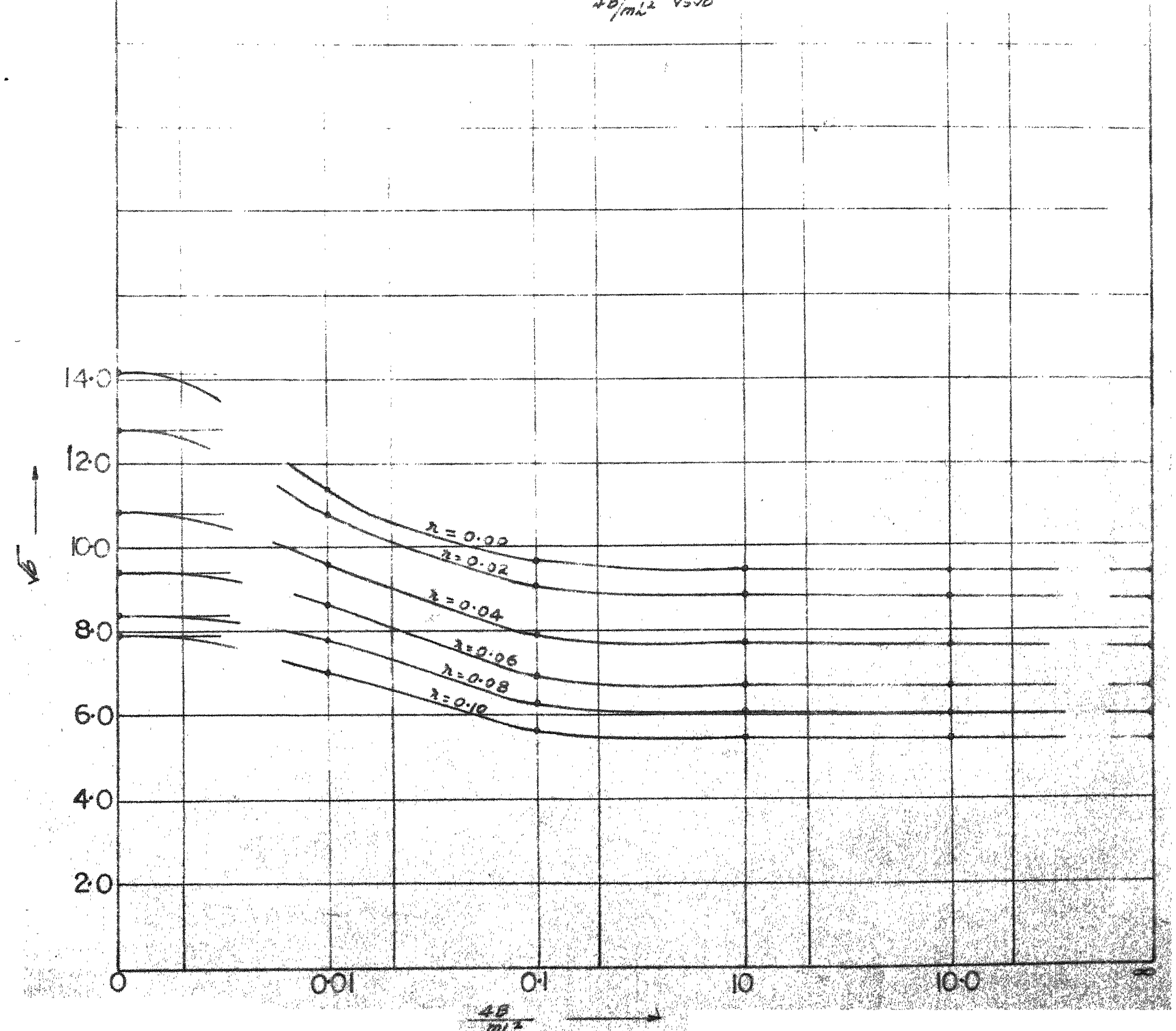

FIG. 6·8

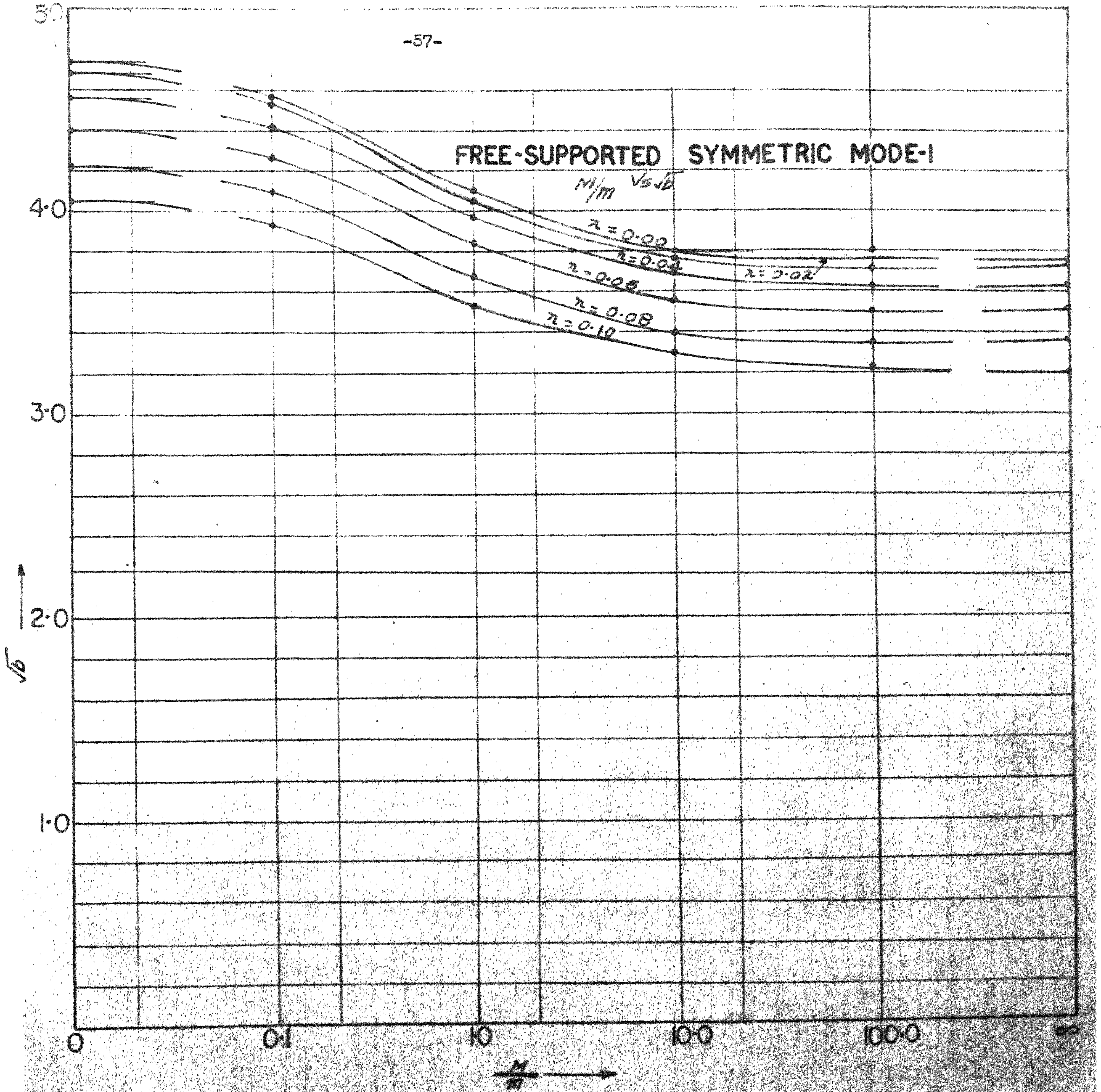
FREE-SUPPORTED SYMMETRIC MODE-I
M/m Vs √b
n = 0.00
n = 0.02
n = 0.04
n = 0.06
n = 0.08
n = 0.10
5.0
4.0
3.0
2.0
1.0
√b
0
0.1
1.0
10.0
100.0
∞
M/m

FIG. 6-9

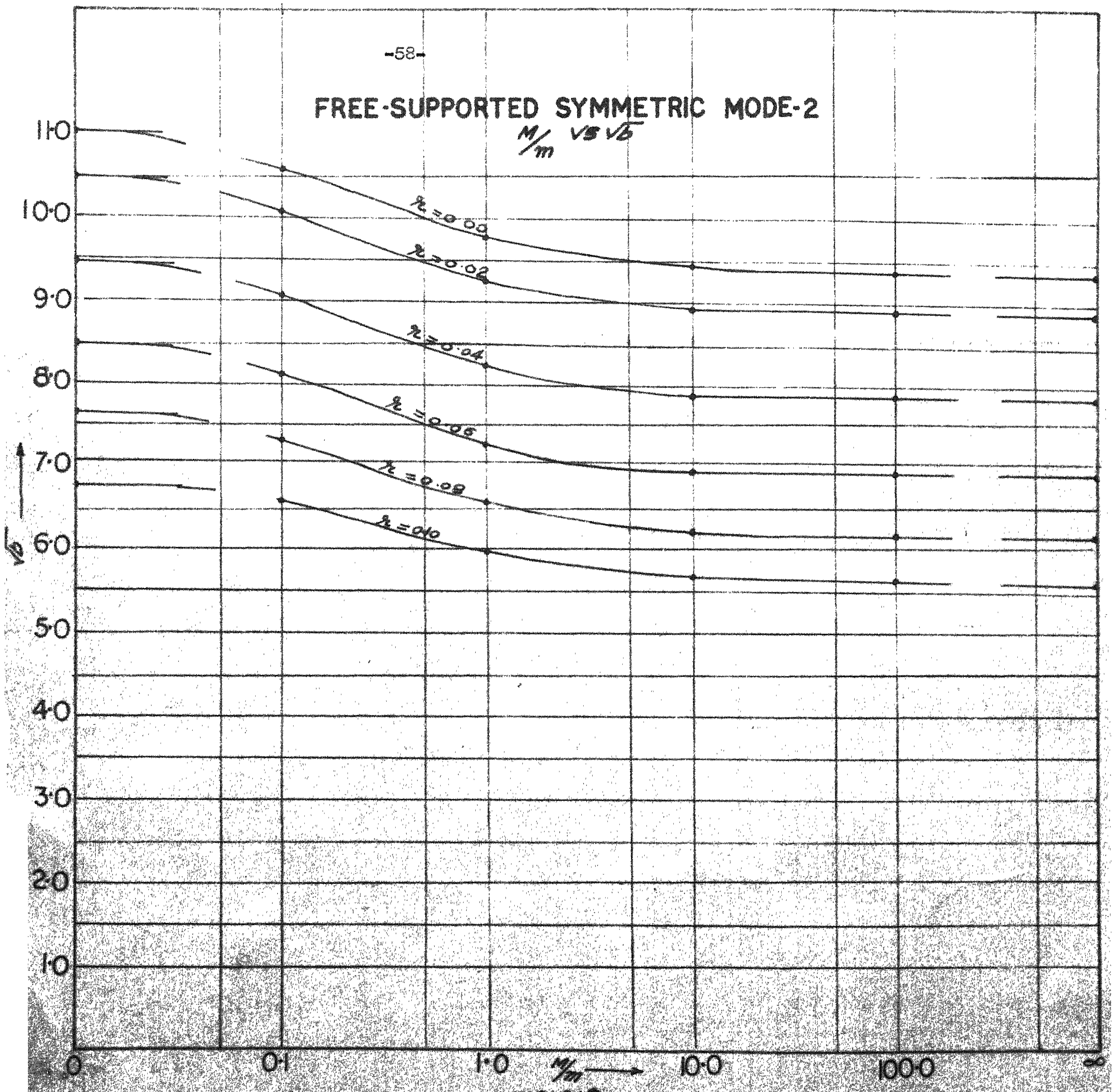
FREE-SUPPORTED SYMMETRIC MODE-2
11·0
10·0
9·0
8·0
7·0
6·0
5·0
4·0
3·0
2·0
1·0
0
0·1
1·0
10·0
100·0
∞

FIG.610

FREE-SUPPORTED ASYMMETRIC MODE-I

$\frac{4B}{mL^2}$ Vs $\sqrt{\sigma}$

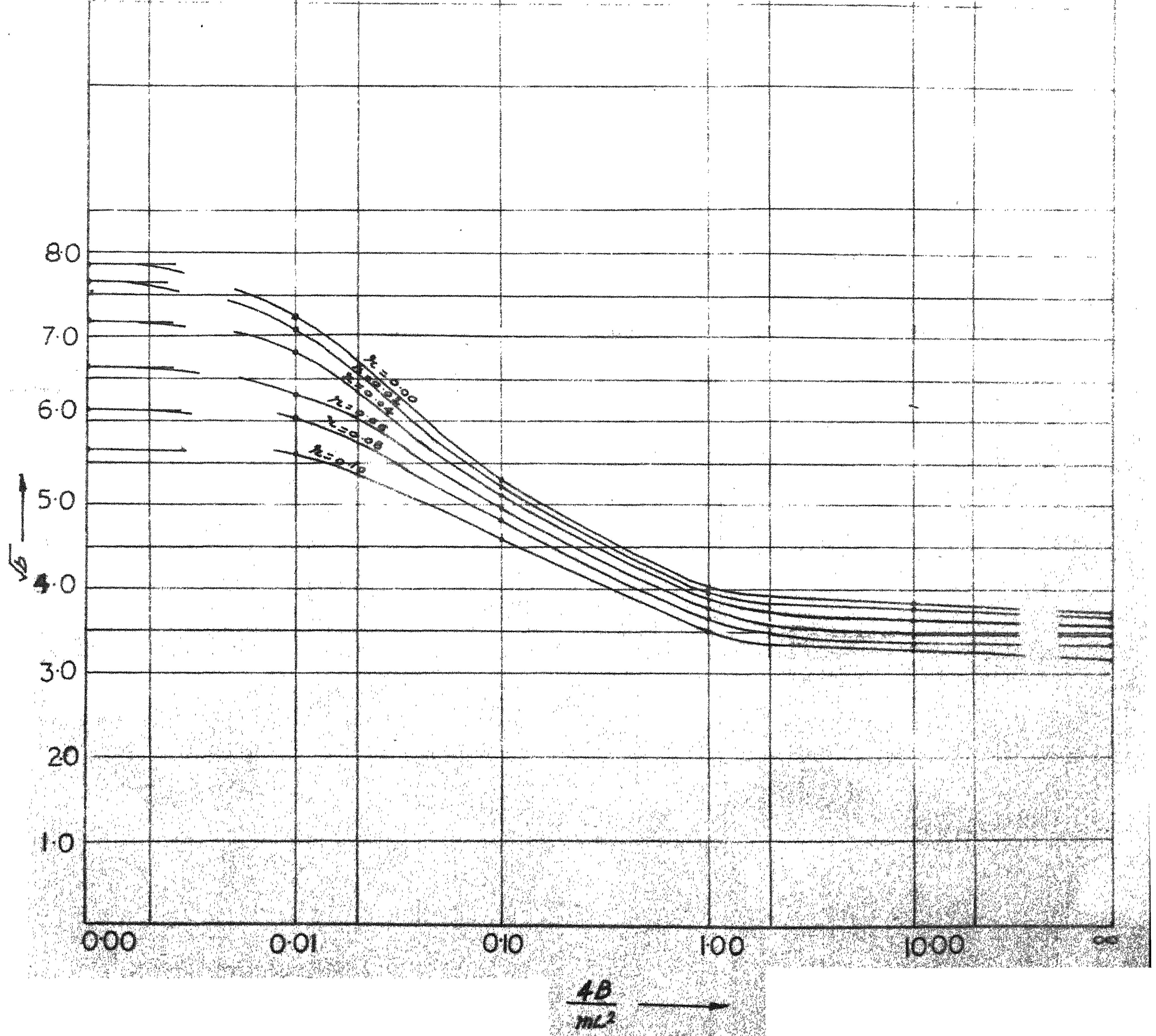

FIG. 6.11.

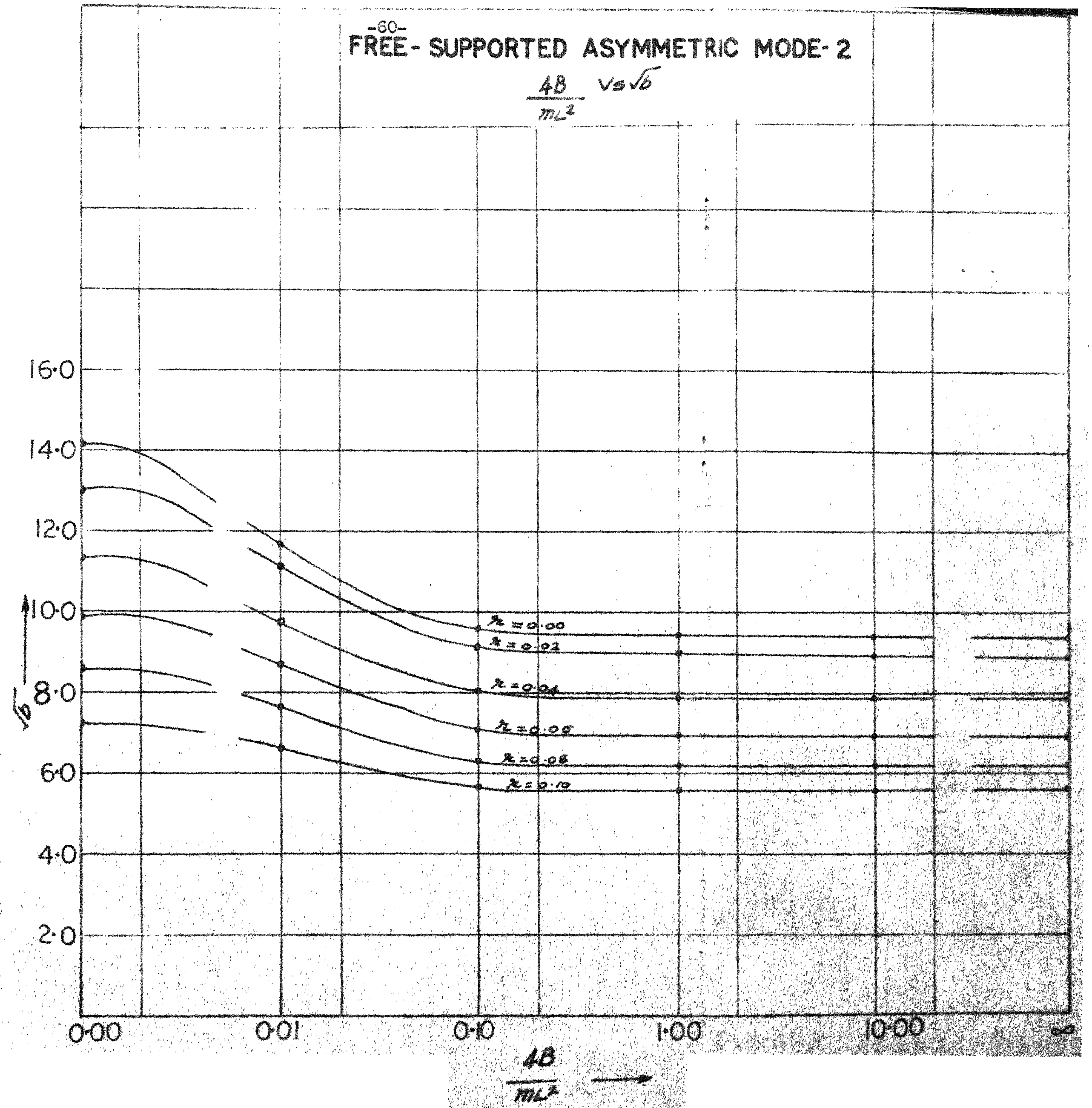
FREE- SUPPORTED ASYMMETRIC MODE- 2
4B/mL² Vs √b
16·0
14·0
12·0
10·0
8·0
6·0
4·0
2·0
√b
r = 0·00
r = 0·02
r = 0·04
r = 0·06
r = 0·08
r = 0·10
0·00
0·01
0·10
1·00
10·00
∞
4B/mL²

FIG. 6·12

CONCLUSIONS

It is observed that by increasing the mass ratio or moment of inertias ratio, frequency decreases for the fixed value of r. It is also observed that for the fixed ratio of masses, as well as inertias, frequency decreases as r increases.

It is also observed for the fixed ratio of masses of the attached mass, and beam, as well as for the fixed values of mass moment of inertias of the attached mass, and the beam, frequency of the composite system decreases as r increases and asymptotically approaches the appropriate limiting cases.

For any given value of r, mass ratio, the ratio of moment of inertias, and boundary conditions frequency can be obtained from the graphs directly.

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